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Functional central limit theorem for Brownian particles in domains with Robin boundary condition <sup>☆</sup>



Zhen-Qing Chen <sup>a,\*</sup>, Wai-Tong (Louis) Fan <sup>b</sup>

<sup>a</sup> Department of Mathematics, University of Washington, Seattle, WA 98195, USA

<sup>b</sup> Department of Mathematics, University of Wisconsin at Madison, Madison, WI 53706, USA

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ABSTRACT

We rigorously derive non-equilibrium space–time fluctuation for the particle density of a system of reflected diffusions in bounded Lipschitz domains in  $\mathbb{R}^d$ . The particles are independent and are killed by a time-dependent potential which is asymptotically proportional to the boundary local time. We generalize the functional analytic framework introduced by Kotelenetz [20,21] to deal with time-dependent perturbations. Our proof relies on Dirichlet form method rather than the machineries derived from Kotelenetz’s sub-martingale inequality. Our result holds for any symmetric reflected diffusion, for any bounded Lipschitz domain and for any dimension  $d \geq 1$ .

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\* Corresponding author.

E-mail addresses: zqchen@uw.edu (Z.-Q. Chen), ariesfanhk@gmail.com (W.-T.(L.) Fan).

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## 1. Introduction

The goal of this paper is to develop a machinery to overcome some difficulties that arise in the study of fluctuations for systems of reflected diffusions (such as reflected Brownian motions) with a singular type of time-dependent killing potential. The primary examples are the systems of annihilating diffusions introduced in [4] and [5], which can be used to model the transport of positive and negative charges in solar cells or the population dynamics of two segregated species under competition. The model in [5] consists of two families of reflected diffusions confined in two adjacent domains, say two adjacent rectangles  $(0, 2) \times (0, 1)$  and  $(0, 2) \times (-1, 0)$ , respectively. These two families of particles (positive and negative charges respectively) annihilate each other at a certain rate when they come close to each other near the interface  $(0, 2) \times \{0\}$ . This interaction models the annihilation, trapping, recombination and separation phenomena of the charges. From the viewpoint of the positive charges, they are themselves reflected diffusions in  $(0, 2) \times (0, 1)$  subject to killing by a time-dependent random potential.

In this paper, we focus our attention to a one-type particle model which consists of i.i.d. reflected diffusions killed by a deterministic time-dependent potential near the boundary. The following assumption on reflected diffusions is in force throughout this paper:

**Assumption 1.1.** Suppose  $D \subset \mathbb{R}^d$  is a bounded Lipschitz domain,  $\rho \in W^{1,2}(D) \cap C(\bar{D})$  is a strictly positive function,  $\mathbf{a} = (a^{ij})$  is a symmetric, bounded, uniformly elliptic  $d \times d$  matrix-valued function such that  $a^{ij} \in W^{1,2}(D)$  for each  $i, j$ . Here  $C(\bar{D})$  denotes the

space of continuous functions on  $\bar{D}$  and  $W^{1,2}(D) := \{f \in L^2(D; dx) : |\nabla f| \in L^2(D; dx)\}$  denotes the usual Sobolev space of order  $(1, 2)$ .

Under [Assumption 1.1](#), it is well known (see [\[1,3\]](#)) that the bilinear form  $(\mathcal{E}, W^{1,2}(D))$  defined by

$$\mathcal{E}(f, g) = \frac{1}{2} \int_D \mathbf{a}(x) \nabla f(x) \cdot \nabla g(x) \rho(x) dx = \frac{1}{2} \int_D \sum_{i,j=1}^d a^{ij}(x) \frac{\partial f}{\partial x_i}(x) \frac{\partial g}{\partial x_j}(x) \rho(x) dx \tag{1.1}$$

is a regular Dirichlet form in  $L^2(\bar{D}, \rho(x)dx)$  and hence has an associated Hunt process  $X$  on  $\bar{D}$  (unique in distribution). Furthermore,  $X$  is a continuous strong Markov process with symmetrizing measure  $\rho$  and has infinitesimal generator  $\mathcal{A} = \frac{1}{2\rho} \nabla \cdot (\rho \mathbf{a} \nabla)$ . Intuitively,  $X$  behaves like a diffusion process associated to the second order elliptic differential operator  $\mathcal{A}$  in the interior of  $D$ , and is instantaneously reflected at the boundary in the inward *conormal* direction  $\vec{\nu} = \mathbf{a}\vec{n}$ , where  $\vec{n}$  is the unit inward normal vector field on  $\partial D$ . See Chen [\[3\]](#) for the Skorokhod representation for  $X$ , which tells us some precise pathwise properties of  $X$ . We call  $X$  an **( $\mathbf{a}, \rho$ )-reflected diffusion** or an **( $\mathcal{A}, \rho$ )-reflected diffusion**. A special but very important case is when  $\mathbf{a}$  is the identity matrix and  $\rho = 1$ , in which  $X$  is called a **reflected Brownian motion (RBM)**. Next, we make the following assumption about the killing potential throughout this paper.

**Assumption 1.2 (Killing potential).** Suppose  $q(t, x)$  is a given non-negative bounded function on  $[0, \infty) \times \bar{D}$  such that  $q(t, \cdot) \in C(\bar{D})$  for all  $t \geq 0$ . Suppose also that  $\delta_N$  is a sequence of positive numbers that converges to zero and denote  $q_N(t, x) = \delta_N^{-1} \mathbf{1}_{D^{\delta_N}}(x)q(t, x)$ , where  $D^\delta = \{x \in D : \text{dist}(x, \partial D) < \delta\}$ .

Our particle system is parameterized by  $N \in \mathbb{N}$ , the initial number of particles. The function  $q_N$  plays the role of a time-dependent killing potential. This killing potential is singular in the sense that  $\delta_N^{-1} \mathbf{1}_{D^{\delta_N}}(x)$  converges weakly to the surface measure  $\sigma$  which is singular with respect to Lebesgue measure. More precisely, for  $N \in \mathbb{N}$ , we let  $\{X_i\}_{i=1}^N$  be independent **( $\mathbf{a}, \rho$ )-reflected diffusions** in  $D$  and  $\{R_i\}_{i=1}^N$  be independent exponential random variables with mean one. The normalized empirical measure of the particles *alive* is defined as:

$$\mathfrak{X}_t^N(dz) := \frac{1}{N} \sum_{\{i: t < \zeta_i^{(N)}\}} \mathbf{1}_{X_i(t)}(dz), \text{ where} \tag{1.2}$$

$$\zeta_i^{(N)} = \inf \left\{ t \geq 0 : \frac{1}{2} \int_0^t q_N(s, X_i(s)) ds \geq R_i \right\}. \tag{1.3}$$

Note that  $\mathfrak{X}_t^N$  is a random measure on  $\bar{D}$ . Moreover,  $\mathfrak{X}^N = (\mathfrak{X}_t^N)_{t \geq 0}$  is a strong Markov process in  $M_+(\bar{D})$ , the space of finite non-negative Borel measures on  $\bar{D}$  equipped with weak topology, and  $\mathfrak{X}^N$  has sample paths in the Skorokhod space  $D([0, \infty), M_+(\bar{D}))$  almost surely.

**Remark 1.3.** Let  $\{Z_i^{(N)}\}_{i=1}^N$  be independent **sub-processes** (cf. [7]) of reflected diffusions killed by the potential  $q_N$ . That is,

$$Z_i^{(N)}(t) := \begin{cases} X_i(t), & t < \zeta_i^{(N)} \\ \partial, & t \geq \zeta_i^{(N)}, \end{cases}$$

where  $\partial$  is an isolated point added to  $\bar{D}$ . Then  $\mathfrak{X}_t^N(dz)$  defined in (1.2) is equal to  $\frac{1}{N} \sum_{i=1}^N \mathbf{1}_{Z_i^{(N)}(t)}(dz)$  if we view  $\mathbf{1}_\partial$  as the zero measure.

We coin this model the name **Robin boundary model** due to the following hydrodynamic result. In what follows,  $\xrightarrow{\mathcal{L}}$  denotes convergence in probability law. That is, for sequence of  $\mathcal{S}$ -valued random variables  $\{X_n; n \geq 1\}$  in a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  where  $\mathcal{S}$  is a Polish space,  $X_n \xrightarrow{\mathcal{L}} X$  if and only if  $\lim_{n \rightarrow \infty} \mathbb{E}[f(X_n)] = \mathbb{E}[f(X)]$  for all  $f \in C_b(\mathcal{S})$ , the space of bounded continuous functions on  $\mathcal{S}$ . Here  $\mathbb{E}$  is the mathematical expectation with respect to the probability measure  $\mathbb{P}$ .

**Theorem 1.4** (Functional law of large numbers). *Suppose Assumptions 1.1 and 1.2 hold. Suppose  $\{\mathfrak{X}_0^N\} \xrightarrow{\mathcal{L}} u_0(x)\rho(x) dx$  in  $M_+(\bar{D})$ , where  $u_0 \in C(\bar{D})$ . Then*

$$\mathfrak{X}_t^N(dx) \xrightarrow{\mathcal{L}} u(t, x)\rho(x) dx \quad \text{in } D([0, \infty), M_+(\bar{D})),$$

where  $u \in C([0, \infty) \times \bar{D})$  is the probabilistic solution<sup>1</sup> to the heat equation  $\frac{\partial u}{\partial t} = \mathcal{A}u$  with Robin boundary condition  $\frac{\partial u}{\partial \bar{\nu}}(t, x) = q(t, x)u(t, x)/\rho(x)$  on  $(0, \infty) \times \partial D$  and initial condition  $u(0, \cdot) = u_0$ .

The proof of Theorem 1.4 is an elementary law of large numbers argument involving the calculation of two moments. Since it is much easier than that of [5], we omit it here and refer the reader to that paper.

### 1.1. Main result

Our object of study in this paper is the **fluctuation process**  $\mathcal{Y}^N = (\mathcal{Y}_t^N)_{t \geq 0}$  defined by

$$\langle \mathcal{Y}_t^N, \phi \rangle := N^{1/2}(\langle \mathfrak{X}_t^N, \phi \rangle - \mathbb{E}\langle \mathfrak{X}_t^N, \phi \rangle) \quad t \geq 0, \phi \in L^2(D), \tag{1.4}$$

<sup>1</sup> By [5],  $u$  has the probabilistic representation  $\mathbb{E}^x [u_0(X_t) \exp(-\int_0^t q(t-s, X_s) dL_s)]$  where  $L_t$  is the boundary local time of  $X$ .

where  $\langle \mathfrak{X}_t^N, \phi \rangle := \frac{1}{N} \sum_{\{i: t < \zeta_i^{(N)}\}} \phi(X_i(t))$  is the integral of an observable (or test function)  $\phi$  with respect to the measure  $\mathfrak{X}_t^N$ . Even in this simple setting, answers to the following natural questions are non-trivial.

- (i) What is the state space for  $\mathcal{Y}_t^N$ ? This space should possess a topology which allows us to make sense of convergence of  $\mathcal{Y}^N$ , if it does converge. Observe that although  $\mathcal{Y}^N$  acts on  $L^2(D)$  linearly, it is not a bounded operator in general.
- (ii) Does  $\mathcal{Y}^N$  converge? If so, what can we say about its limit?

The answer for question (i) is given by Lemma 4.1. It says that the process  $(\mathcal{Y}_t^N)_{t \geq 0}$  has sample paths in  $D([0, \infty), \mathcal{H}_{-\alpha})$  for  $\alpha > 0$  large enough, where  $\mathcal{H}_{-\alpha}$  is a Hilbert space of distributions that strictly contains  $L^2(D, \rho(x)dx)$ . See Subsection 2.2 for the precise construction of  $\mathcal{H}_{-\alpha}$ , which can be identified with the dual of the Sobolev space  $W^{\alpha/2, 2}(D)$  of fractional order.

The answer for question (ii) is given by Theorem 1.5, the main result of this paper. Theorem 1.5 contains 2 parts: the convergence result and the properties of the limit. The limit is shown to be decomposable into an independent sum of a “transportation part” and a “white noise part” (see (1.7) below). The ‘transportation part’ is governed by the evolution operators  $\{Q_{s,t}\}_{s \leq t}$  generated on  $C(\bar{D})$  by the backward PDE  $\frac{\partial v}{\partial s} = -\mathcal{A}v$  on  $(0, t) \times D$  with Robin boundary condition  $\frac{\partial v}{\partial \nu} = qv/\rho$  on  $(0, t) \times \partial D$ . More precisely, for  $0 \leq s \leq t$  and  $\phi \in L^2(D)$ , we define

$$\begin{aligned}
 Q_{s,t}\phi(x) &:= \mathbb{E} \left[ \phi(X_t) \exp \left( - \int_s^t q(r, X_r) dL_r \right) \middle| X_s = x \right] \\
 &= \mathbb{E} \left[ \phi(X_{t-s}) \exp \left( - \int_0^{t-s} q(s+r, X_r) dL_r \right) \middle| X_0 = x \right]. \tag{1.5}
 \end{aligned}$$

Define

$$\mathbf{U}_{(t,s)}\mu(\phi) := \mu(Q_{s,t}\phi) \tag{1.6}$$

for  $\alpha > 0$ ,  $\mu \in \mathcal{H}_{-\alpha}$  and  $\phi \in L^2(D)$  whenever it is well defined (i.e.  $Q_{s,t}\phi \in \mathcal{H}_\alpha$ ); see Theorem 1.5 and Remark 2.1. For simplicity, denote by  $\langle \phi, \psi \rangle_\rho := \int_D \phi(x)\psi(x)\rho(x)dx$  the inner product of  $L^2(D, \rho(x)dx)$ .

Hereafter we will use  $\stackrel{\mathcal{L}}{=}$  to denote equal in probability law. The following is the main result of this paper.

**Theorem 1.5** (Functional central limit theorem). *Suppose that Assumptions 1.1 and 1.2 hold and that the initial positions of particles are i.i.d with distribution  $u_0(x)\rho(x)dx$ , where  $u_0 \in C(\bar{D})$ . Then for any  $\alpha > d + 2$  and  $T > 0$ ,  $\mathcal{Y}^N$  converges to  $\mathcal{Y}$  in distribution as  $N \rightarrow \infty$  in the Skorokhod space  $D([0, T], \mathcal{H}_{-\alpha})$ , where  $\mathcal{Y}$  is the generalized*

Ornstein–Uhlenbeck process taking values in  $D([0, T], \mathcal{H}_{-\alpha})$  given by

$$\mathcal{Y}_t \stackrel{\mathcal{L}}{=} \mathbf{U}_{(t,0)} \mathcal{Y}_0 + \int_0^t \mathbf{U}_{(t,s)} dM_s. \tag{1.7}$$

In the above,  $M$  is a (unique in distribution) continuous,  $\tilde{\mathcal{F}}_t$ -adapted, square integrable,  $\mathcal{H}_{-\alpha}$ -valued Gaussian martingale with independent increments and covariance functional characterized by

$$\tilde{\mathbb{E}} [\langle M_t, \phi \rangle^2] = \int_0^t \left( \langle \mathbf{a} \nabla \phi \cdot \nabla \phi, u(s) \rangle_\rho + \int_{\partial D} \phi^2(z) u(s, z) q(s, z) \rho(z) d\sigma(z) \right) ds, \quad \phi \in \mathcal{H}_\alpha, \tag{1.8}$$

defined on a complete probability space with right continuous filtration  $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathcal{F}}_t, \tilde{\mathbb{P}})$ , where the function  $u(s, x)$  is given by [Theorem 1.4](#).  $\mathcal{Y}_0$  is the centered Gaussian random variable with covariance

$$\tilde{\mathbb{E}} [\mathcal{Y}_0(\phi) \mathcal{Y}_0(\psi)] = \langle \phi \psi, u_0 \rangle_\rho - \langle \phi, u_0 \rangle_\rho \langle \psi, u_0 \rangle_\rho \quad \text{for } \phi, \psi \in \mathcal{H}_\alpha,$$

defined on the same probability space as  $M$  and is independent of  $M$ . Moreover,  $\mathcal{Y}$  is a continuous Gaussian Markov process which is unique in distribution, and  $\mathcal{Y}$  has a version in  $C^\gamma([0, \infty), \mathcal{H}_{-\alpha})$  (i.e. Hölder continuous with exponent  $\gamma$ ) for any  $\gamma \in (0, 1/2)$ .

**Remark 1.6.**

- (i) In [\(1.7\)](#),  $\int_0^t \mathbf{U}_{(t,s)} dM_s$  is the stochastic integral with respect to the Hilbert space valued martingale  $M$  (cf. [\[23\]](#)). In [Appendix A](#), we prove that it is well-defined. For the convenience of the reader, we also state the precise definition of Hilbert space valued continuous Gaussian processes with independent increment. The existence and uniqueness of  $M$  are given in [Theorem 4.6](#). Furthermore, for  $\alpha > d + 2$ , both  $\mathbf{U}_{(t,0)} \mathcal{Y}_0$  and  $\int_0^t \mathbf{U}_{(t,s)} dM_s$  live in  $\mathcal{H}_{-\alpha}$  (i.e. they extend to be continuous functionals on  $\mathcal{H}_\alpha$ ).
- (ii) Roughly speaking,  $\mathcal{Y}$  solves the following stochastic evolution equation (called the **Langevin equation**) in the weak sense:

$$dY_t = \mathbf{A}_t^{(-\alpha)} Y_t dt + dM_t, \quad Y_0 = \mathcal{Y}_0, \tag{1.9}$$

where  $\mathbf{A}_t^{(-\alpha)}$  is the generator of  $\{\mathbf{U}_{(t,s)}\}_{t \geq s}$  in the Hilbert space  $\mathcal{H}_{-\alpha}$ .

- (iii) Define a bilinear forms  $\mathcal{E}_s^{(q)}$  on  $L^2(D, \rho(x)dx) \cap L^2(\partial D, d\sigma)$  by

$$\mathcal{E}_s^{(q)}(\phi, \psi) := \langle \mathbf{a} \nabla \phi \cdot \nabla \psi, u(s) \rangle_\rho + \int_{\partial D} \phi \psi u(s) q(s) \rho d\sigma \tag{1.10}$$

and  $\mathcal{E}_s^{(q)}(\phi) := \mathcal{E}_s^{(q)}(\phi, \phi)$  for  $s \geq 0$ . Now (1.8) reads as  $\mathbb{E}[\langle M_t, \phi \rangle^2] = \int_0^t \mathcal{E}_s^{(q)}(\phi) ds$ . As an immediate application of (1.7), for all fixed  $\phi \in \mathcal{H}_\alpha$  with  $\alpha > d + 2$ , we have

$$\mathcal{Y}_t(\phi) \stackrel{\mathcal{L}}{=} \mathcal{Y}_0(Q_{0,t}\phi) + \int_0^t \sqrt{\mathcal{E}_s^{(q)}(Q_{s,t}\phi)} dB_s^{(\phi)} \quad \text{in } D([0, T], \mathbb{R}), \tag{1.11}$$

where  $B^{(\phi)}$  is a standard Brownian motion independent of  $\mathcal{Y}_0$ . Therefore, we can simulate the evolution (in time  $t$ ) of the fluctuations of the particle density with respect to an observable  $\phi$  by running a Brownian motion.

- (iv) When  $D$  is a cube (such as when  $d = 1$ ), Theorem 1.5 holds with  $\alpha > d/2 + 2$  in place of  $\alpha > d + 2$ , since we have a stronger uniform upper bound for eigenfunctions, namely  $\sup_\ell \|\phi_\ell\| < C(d, D)$ .  $\square$

**Remark 1.7.**

- (i) When  $q = 0$ , Theorem 1.5 in particular gives the fluctuation result for independent reflecting Brownian motions in bounded Lipschitz domains.
- (ii) (Killing by local time) Clearly, the measure  $q_N(t, x) dx$  converges weakly to  $q(t, x) d\sigma(x)$  as  $N \rightarrow \infty$ , where  $\sigma$  denotes the surface measure on  $\partial D$ . The positive additive continuous functional (see the Appendix of [7]) of  $X_i$  having Revuz measure  $q(t, x) d\sigma(x)$  is  $2 \int_0^t q(s, X_i(s)) dL_s^{(i)}$ , where  $L_t^{(i)}$  is the boundary local time of  $X_i$ . Hence it is natural to ask: what if the processes  $\{X_i\}_{i \geq 1}$  are killed by  $2 \int_0^t q(s, X_i(s)) dL_s^{(i)}$  (which no longer depends on  $N$ ) rather than by a potential function  $q_N$  on the strip  $D^{\delta_N}$ ? It turns out that, with little extra effort, one can show that Theorem 1.4 and Theorem 1.5 both remain valid if we replace the definition of  $\zeta_i^{(N)}$  in (1.3) by

$$\zeta_i^{(N)} = \zeta_i := \inf \left\{ t \geq 0 : 2 \int_0^t q(s, X_i(s)) dL_s^{(i)} \geq R_i \right\}. \tag{1.12}$$

See Subsection A.4 for details.  $\square$

One of the earliest rigorous results about fluctuation limits was proven by Itô [16,17], who considered a system of independent and identically distributed (i.i.d.) Brownian motions in  $\mathbb{R}^d$  and showed that the limit is an  $\mathcal{S}'$ -valued Gaussian process solving a Langevin equation, where  $\mathcal{S}'$  is the Schwartz space of tempered distributions. Fluctuation limits for stochastic particle systems in domains are very limited. Sznitman [28] studied the fluctuations of a conservative system of diffusions with normal reflected boundary conditions on smooth domains. Fluctuations of the reaction–diffusion systems on the cube  $[0, 1]^d$  with linear or quadratic reaction terms were studied in [20,10,21,2]. These fluctuation

results are valid only for dimension  $d \leq 3$ . Roughly speaking, for the reaction–diffusion equation

$$\frac{\partial u}{\partial t} = \frac{1}{2} \Delta u + R(u)$$

where  $R(u)$  is a polynomial reaction term, the transportation component of the fluctuations is shown to satisfy  $\frac{\partial v}{\partial t} = \frac{1}{2} \Delta v + R'(u)v$ . Here  $R'(u)$  is the derivative of  $R$  (e.g.,  $R'(u) = 2u$  if  $R(u) = u^2$ ). That is, the fluctuation limit  $\mathcal{Y}$  weakly solves the stochastic partial differential equation (called a **Langevin equation**)

$$d\mathcal{Y}_t = \left( \frac{1}{2} \Delta \mathcal{Y}_t + R'(u(t))\mathcal{Y}_t \right) dt + dM_t,$$

where  $R'(u)$  is viewed as a multiplicative operator and  $M$  is a Gaussian martingale. In a subsequent paper [6], we study the fluctuation limit for an interacting diffusion system of two species introduced in [5].

### 1.2. Outline of proof

We prove [Theorem 1.5](#) through the following six steps.

Step 1:  $\mathcal{Y}^N$  satisfies the following stochastic integral equation

$$\mathcal{Y}_t^N = \mathbf{U}_{(t,0)}^N \mathcal{Y}_0^N + \int_0^t \mathbf{U}_{(t,s)}^N dM_s^N \text{ a.s.,}$$

where  $\mathbf{U}_{(t,s)}^N$  is an evolution system approximating  $\mathbf{U}_{(t,s)}$ ; see [Theorem 4.3](#).

Step 2:  $M^N \xrightarrow{\mathcal{L}} M$  in  $D([0, T], \mathcal{H}_{-\alpha})$ ; see [Theorem 4.6](#).

Step 3:  $\mathcal{Y}^N$  is tight in  $D([0, T], \mathcal{H}_{-\alpha})$ ; see [Theorem 4.7](#).

Step 4:  $\mathbf{U}_{(t,0)}^N \mathcal{Y}_0^N \xrightarrow{\mathcal{L}} \mathbf{U}_{(t,0)} \mathcal{Y}_0$  in  $D([0, T], \mathcal{H}_{-\alpha})$ ; see [Theorem 4.8](#).

Step 5:  $\int_0^t \mathbf{U}_{(t,s)}^N dM_s^N \xrightarrow{\mathcal{L}} \int_0^t \mathbf{U}_{(t,s)} dM_s$  in  $D([0, T], \mathcal{H}_{-\alpha})$ ; see [Theorem 4.9](#).

Step 6: All the stated properties for the fluctuation limit hold; see [Theorem 4.11](#).

The main difficulty is in establishing the convergence in Step 5. Note that  $t \mapsto \int_0^t \mathbf{U}_{(t,s)} dM_s$  is not a martingale. The standard method based on Kotelenetz’s submartingale inequality [19] does not seem to work. This is because in our case  $\mathbf{U}_{(t,s)}$  is not exponentially bounded; that is, there is no  $\beta > 0$  so that the operator norm  $\|\mathbf{U}_{(t,s)}\| \leq e^{\beta(t-s)}$  for  $t \geq s$  (see [19]). In fact, we suspect it is not even a bounded operator on  $\mathcal{H}_{-\alpha}$  due to the singular interaction near the boundary. To overcome this difficulty, we need first to make sense of the expression  $\int_0^t \mathbf{U}_{(t,s)} dM_s$ , which is done in Section 4, [Appendix A](#) of this paper. Our approach is then based on suitably extending



the functional analytic framework of [20] and a direct analysis that uses heat kernel estimates and Dirichlet Form method.

## 2. Functional analytic framework

Our method to study the fluctuation is functional analytic, with the mathematical framework being the calculus of evolution equations on Hilbert spaces (see, for example, [9,13,15]). As remarked in [20], this approach yields a useful representation of the limiting process (the generalized Ornstein–Uhlenbeck process) as the mild solution of a stochastic partial differential equation (SPDE), which yields uniqueness and Gaussian property for free. It also tells us the smallest Hilbert space in which the generalized Ornstein–Uhlenbeck process lives.

### Conventions and notations:

In this paper, we use  $:=$  as a way of definition. For  $a, b \in \mathbb{R}$ ,  $a \vee b := \max\{a, b\}$  and  $a \wedge b := \min\{a, b\}$ . We use abbreviation r.c.l.l. for right continuous having left limits, and  $\|\cdot\|$  to denote the supremum norm in  $\bar{D}$ . Even though the constants appearing in the article may depend on  $\mathbf{a}$  or  $\rho$  given in Assumption 1.1, we will not mention this dependence explicitly. For example, we use  $C(d, D)$  to denote a constant which depends only on  $d$  and  $D$  (and possibly on  $\mathbf{a}$  or  $\rho$ ). The exact value of the constant may vary from line to line.

### 2.1. Neumann heat kernel

It is well known (cf. [1,14] and the references therein) that, on a bounded Lipschitz domain  $D$ , an  $(\mathcal{A}, \rho)$ -reflected diffusion  $X$  has a jointly locally Hölder continuous transition density  $p(t, x, y)$  with respect to the symmetrizing measure  $\rho(x)dx$  on  $(0, \infty) \times \bar{D} \times \bar{D}$ . Moreover, the following Aronson-type Gaussian estimates hold:

$$\frac{1}{c_1 t^{d/2}} \exp\left(\frac{-c_2|y-x|^2}{t}\right) \leq p(t, x, y) \leq \frac{c_1}{t^{d/2}} \exp\left(\frac{-|y-x|^2}{c_2 t}\right) \tag{2.1}$$

for every  $(t, x, y) \in (0, T] \times \bar{D} \times \bar{D}$ , where  $c_1 = c_1(d, D, T)$  and  $c_2 = c_2(d, D, T)$  are positive finite constants.

Using (2.1) and the Lipschitz assumption for  $\partial D$ , we can check that for all  $T > 0$ ,

$$\sup_{\varepsilon \in (0, \varepsilon_0)} \sup_{x \in \bar{D}} \frac{1}{\varepsilon} \int_{D^\varepsilon} p(t, x, y) dy \leq \frac{C}{t^{1/2}} \quad \text{for } t \in (0, T], \tag{2.2}$$

where  $\varepsilon_0 = \varepsilon_0(D) \in (0, \infty)$  and  $C = C(d, D, T) \in (0, \infty)$ . In particular, we can let  $\varepsilon \rightarrow 0$  in (2.2) to obtain, via (3.1),

$$\sup_{x \in \bar{D}} \int_{\partial D} p(t, x, y) d\sigma(y) \leq \frac{C}{t^{1/2}} \quad \text{for } t \in (0, T]. \tag{2.3}$$

2.2. Hilbert space  $\mathcal{H}_\gamma$

Recall that  $\mathcal{A} = \frac{1}{2\rho} \nabla \cdot (\rho \mathbf{a} \nabla)$  denotes the  $L^2(D, \rho(x)dx)$ -generator for an  $(\mathbf{a}, \rho)$ -reflected diffusion. Clearly,  $\mathcal{A}$  is a self-adjoint, non-positive operator on  $L^2(D, \rho(x)dx)$ . Together with the fact that  $D$  is bounded, we see that  $\mathcal{A}$  has a discrete spectrum in  $\mathcal{H}_0$ . Let  $\phi_k$  be a complete orthonormal system (CONS) of eigenvectors of  $\mathcal{A}$  in  $\mathcal{H}_0$  with eigenvalues  $-\lambda_k$ , where  $0 = \lambda_1 < \lambda_2 \leq \lambda_3 \leq \dots$ . Note that the linear span of  $\{\phi_k\}$  is dense in  $L^2(D; \rho dx)$ . We define, for  $\alpha \in (-\infty, \infty)$ ,

$$\mathcal{H}_\alpha := \text{the closure of the linear span of } \{\phi_k\} \text{ with respect to } \langle \cdot, \cdot \rangle_\alpha, \tag{2.4}$$

where  $\langle \phi, \psi \rangle_\alpha := \langle (I - \mathcal{A})^\alpha \phi, \psi \rangle_\rho$ . Here  $I$  is the identity operator on  $\mathcal{H}_0 = L^2(D; \rho dx)$  and  $(I - \mathcal{A})^\alpha$  is the  $\alpha$ th power (defined through spectral representation) of the positive self-adjoint operator  $I - \mathcal{A}$ . In particular,  $\langle \cdot, \cdot \rangle_0 = \langle \cdot, \cdot \rangle_\rho$  by definition.

Note that  $(\mathcal{H}_\alpha, \langle \cdot, \cdot \rangle_\alpha)$  is a real separable Hilbert space and that  $\mathcal{H}_\beta \subset \mathcal{H}_\alpha$  when  $\beta > \alpha$ . Moreover,  $\mathcal{H}_\alpha$  and  $\mathcal{H}_{-\alpha}$  are dual to each other. Equip  $\Phi := \bigcap_{\alpha \geq 0} \mathcal{H}_\alpha$  with the locally convex topology defined by the set of norms  $\{|\varphi|_\alpha := \langle \varphi, \varphi \rangle_\alpha^{1/2} : \varphi \in \Phi, \alpha \in [0, \infty)\}$ . Let  $\Phi'$  be the strong dual of  $\Phi$ . Identifying  $\mathcal{H}_0$  with its dual  $\mathcal{H}'_0$ , we obtain the chain of dense continuous inclusions

$$\Phi \subset \mathcal{H}_\alpha \subset \mathcal{H}_0 = \mathcal{H}'_0 \subset \mathcal{H}_{-\alpha} \subset \Phi', \quad \alpha \in [0, \infty). \tag{2.5}$$

Moreover, for  $\beta \in \mathbb{R}$ , we have

$$h_k^{(\beta)} := (1 + \lambda_k)^{-\beta/2} \phi_k \quad \text{is a CONS for } \mathcal{H}_\beta. \tag{2.6}$$

Hence,  $\langle \phi, \psi \rangle_\beta = \sum_{k \geq 1} \langle \phi, h_k^{(-\beta)} \rangle \langle \psi, h_k^{(-\beta)} \rangle$  for  $\phi, \psi \in \mathcal{H}_\beta$  and

$$\mathcal{H}_\beta = \left\{ \mu \in \Phi' : \sum_{k \geq 1} \langle \mu, h_k^{(-\beta)} \rangle^2 < \infty \right\}, \tag{2.7}$$

where  $\langle \cdot, \cdot \rangle$  denotes the dual pairing extending  $\langle \cdot, \cdot \rangle_\rho$ .

**Remark 2.1.** When  $\alpha > 0$ ,  $\mathcal{H}_\alpha$  can be identified with the fractional Sobolev space  $W^{\alpha/2,2}(D)$  on  $D$ . This is because for  $\alpha \geq 0$ ,  $\mathcal{H}_\alpha = (I - \mathcal{A})^{-\alpha/2} L^2(D, \rho dx)$ . Since  $D$  is a bounded Lipschitz domain, it is known that  $\mathcal{H}_\alpha = W^{\alpha/2,2}(D)$  when  $\alpha = 1$  (see [3]) and hence for every integer  $\alpha \geq 1$ . It follows by interpolation that  $\mathcal{H}_\alpha = W^{\alpha/2,2}(D)$  for every  $\alpha > 0$ . When  $\alpha < 0$ ,  $\mathcal{H}_\alpha$  can be identified as the dual space of  $\mathcal{H}_{-\alpha}$ .  $\square$

2.3. Weyl’s law and eigenfunction estimates

For a general bounded Lipschitz domain  $D \subset \mathbb{R}^d$ , the Weyl’s asymptotic law for the Neumann eigenvalues holds (see [24]). That is, the number of eigenvalues (counting their

multiplicities) less than or equal to  $x$ , denoted by  $\#\{k : \lambda_k \leq x\}$ , satisfies

$$\lim_{x \rightarrow \infty} \frac{\#\{k : \lambda_k \leq x\}}{x^{d/2}} = C \quad \text{for some constant } C = C(d, D) \in (0, \infty). \tag{2.8}$$

From now on, we denote by  $\|\cdot\|$  the supremum norm. Denote by  $\{P_t; t \geq 0\}$  the transition semigroup of an  $(\mathbf{a}, \rho)$ -reflected diffusion in  $L^2(D, \rho(x)dx)$  (i.e.  $P_t f(x) = \mathbb{E}^x[f(X_t)] = \int_D f(y)p(t, x, y)\rho(y) dy$ ).

**Lemma 2.2.** *There exists  $C = C(d, D) > 0$  such that for all integers  $k \geq 1$  we have*

$$\|\phi_k\| \leq C \lambda_k^{d/4} \quad \text{and} \quad \int_{\partial D} \phi_k^2 d\sigma \leq C (\lambda_k + 1). \tag{2.9}$$

**Proof.** By Cauchy–Schwartz inequality, Chapman–Kolmogorov equation and then the Gaussian upper bound, we have

$$\begin{aligned} |\phi_k(x)| &= e^{\lambda_k t} |P_t \phi_k(x)| \leq e^{\lambda_k t} \|\phi_k\|_{L^2(\rho)} \sqrt{p(2t, x, x)} \\ &\leq e^{\lambda_k t} \frac{C(d, D)}{t^{d/4}} \quad \text{for } t \leq 1/\lambda_1. \end{aligned}$$

Taking  $t = 1/\lambda_k$  yields the first inequality in (2.9).

Recall that the Dirichlet form  $(\mathcal{E}, \text{Dom}(\mathcal{E}))$  (in  $L^2(D, \rho(x)dx)$ ) for the  $(\mathcal{A}, \rho)$ -reflected diffusion  $X$  is regular (since  $D$  has Lipschitz boundary (cf. [1])) and that the surface measure  $\sigma$  is smooth. Hence by Theorem 2.1 of [27], we have the following generalized trace theorem:

$$\int_{\partial D} f(x)^2 \sigma(dx) \leq \|G_\beta \sigma\| \left( \mathcal{E}(f, f) + \beta \int_D f^2(x) dx \right) \tag{2.10}$$

for any  $f \in \text{Dom}(\mathcal{E})$  and  $\beta > 0$ , where  $G_\beta \sigma(x) := \int_0^\infty \int_{\partial D} e^{-\beta t} p(t, x, y) \sigma(dy) dt$ . Note that  $\|G_\beta \sigma\| < \infty$  by (2.3) and the fact that  $p(t, x, y)$  converges to  $1/\int_D \rho(x)dx$  as  $t \rightarrow \infty$  uniformly for  $(x, y) \in \bar{D} \times \bar{D}$  exponentially fast (by eigenfunction expansion). Hence, taking  $\beta = 1$ , we obtain the second inequality in (2.9).  $\square$

### 3. Preliminaries

#### 3.1. Minkowski content for $\partial D$

By the same proof of [5, Lemma 7.1], we obtain the following result.

**Lemma 3.1.** *Let  $D \subset \mathbb{R}^d$  be a bounded Lipschitz domain and  $k \in \mathbb{N}$ . If  $\mathcal{F} \subset C(\overline{D}^k)$  is an equi-continuous and uniformly bounded family of functions, then*

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon^k} \int_{(D^\varepsilon)^k} f(z_1, \dots, z_k) dz_1 \cdots dz_k = \int_{(\partial D)^k} f(z_1, \dots, z_k) \sigma(dz_1) \cdots \sigma(dz_k)$$

uniformly for  $f \in \mathcal{F}$ , where  $D^\varepsilon := \{x \in D : \text{dist}(x, \partial D) < \varepsilon\}$  is the  $\varepsilon$ -neighborhood of  $\partial D$  in  $D$  and  $\sigma$  is the surface measure on  $\partial D$ .

By a simple modification of the same proof, we can strengthen the above lemma as follows.

**Lemma 3.2.** *Let  $D \subset \mathbb{R}^d$  be a bounded Lipschitz domain,  $I$  be a  $\mathcal{H}^{d-1}$ -rectifiable closed subset of  $\partial D$  and  $k \in \mathbb{N}$ . If  $\mathcal{F} \subset \mathcal{B}(D^k)$  is an equi-continuous and uniformly bounded family of functions on an open neighborhood of  $I^k$ , then*

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon^k} \int_{(I^\varepsilon)^k} f(z_1, \dots, z_k) dz_1 \cdots dz_k = \int_{I^k} f(z_1, \dots, z_k) \sigma(dz_1) \cdots \sigma(dz_k)$$

uniformly for  $f \in \mathcal{F}$ , where  $I^\varepsilon := \{x \in D : \text{dist}(x, I) < \varepsilon\}$  is the  $\varepsilon$ -neighborhood of  $I$  in  $D$ .

The following is about a convergence result uniform in the shrinking rate of  $\delta = \delta_N$ . It is used to guarantee that  $\delta_N$  can be any sequence (which converges to zero) in the proof of [Lemma 4.10](#).

**Lemma 3.3.** *Suppose  $\{\mathfrak{X}_0^N\} \xrightarrow{\mathcal{L}} u_0(x)\rho(x) dx$  in  $M_+(\overline{D})$  as  $N \rightarrow \infty$ , where  $u_0 \in C(\overline{D})$ . Let  $\{\varphi_N(r) : r \geq 0, N \in \mathbb{N}\}$  be a family of non-negative continuous functions on  $\overline{D}$  such that  $\sup_N \sup_{r \geq 0} \|\varphi_N(r)\| < \infty$ . For any  $\delta_N \rightarrow 0, T > 0$  and  $p \geq 1$ , we have*

$$\lim_{N \rightarrow \infty} \mathbb{E} \left[ \left( \sup_{t \in [0, T]} \left| \int_0^t \langle \varphi_N(r) \delta_N^{-1} \mathbf{1}_{D^{\delta_N}}, \mathfrak{X}_r^N \rangle - \langle \varphi_N(r) \delta_N^{-1} \mathbf{1}_{D^{\delta_N}}, u(r) \rangle_\rho dr \right|^p \right) \right] = 0, \tag{3.1}$$

where  $\mathbf{1}_{D^{\delta_N}}$  is the indicator function on  $D^{\delta_N}$ .

**Proof.** Let  $H_N(t) := \int_0^t \langle \varphi_N(r) \delta_N^{-1} \mathbf{1}_{D^{\delta_N}}, \mathfrak{X}_r^N \rangle dr$  and  $G_N(t) := \int_0^t \langle \varphi_N(r) \delta_N^{-1} \mathbf{1}_{D^{\delta_N}}, u(r) \rangle_\rho dr$ . It can be shown, by a standard argument and using [Lemma 3.1](#), that for any  $T > 0$ ,

$$H_N(t) - G_N(t) \xrightarrow{\mathcal{L}} 0 \quad \text{in } C([0, T], \mathbb{R}).$$

In particular, by the metric of  $C([0, T], \mathbb{R})$  and the deterministic nature of the limit, we have

$$\sup_{t \in [0, T]} |H_N(t) - G_N(t)| \rightarrow 0 \quad \text{both in law and in probability.}$$

On other hand, since  $H_N(t)$  and  $G_N(t)$  are increasing, we have

$$\limsup_{N \rightarrow \infty} \mathbb{E} \left[ \left( \sup_{t \in [0, T]} |H_N(t) - G_N(t)| \right)^p \right] \leq \limsup_{N \rightarrow \infty} 2^p \left( \mathbb{E}[H_N^p(T)] + G_N^p(T) \right).$$

Furthermore, we can check that  $\limsup_{N \rightarrow \infty} \mathbb{E}[H_N^p(T) + G_N^p(T)] < \infty$ . Denote by  $\mathcal{P}(\bar{D})$  the collection of sub-probability measures on  $\bar{D}$ . Comparing with the process without killing (i.e. replacing the sub-processes  $Z_i^{(N)}$  by the reflected diffusions  $X_i$  in the definition of  $\mathfrak{X}^N$ ), we have by (2.2)

$$\begin{aligned} \sup_{\mu \in \mathcal{P}(\bar{D})} \mathbb{E}_\mu [H_N(t)] &\leq \|\varphi_N\| \sup_{x \in \bar{D}} \mathbb{E}_x \int_0^t \mathbf{1}_{D^{\delta_N}}(X_1(r)) dr \\ &= \|\varphi_N\| \sup_{x \in \bar{D}} \int_0^t \int_{D^{\delta_N}} p(r, x, y) dy dr \leq C_1 t^{1/2}, \end{aligned}$$

where  $C_1$  is a positive constant independent of  $N$  and  $t$ . Let  $f(r) := \langle \delta_N^{-1} \mathbf{1}_{D^{\delta_N}}, \mathfrak{X}_r^N \rangle$ . Then for any positive integer  $k$ , by Fubinni’s theorem and the Markov property, we have for any initial distribution  $\mu$  of  $\mathfrak{X}_0^N$ ,

$$\mathbb{E}_\mu [H_N^k(T)] = k! \mathbb{E} \int_{0 \leq r_1 \leq r_2 \leq \dots \leq r_k \leq T} f(r_1) f(r_2) \dots f(r_k) dr_1 dr_2 \dots dr_k \leq k! (C_1 T^{1/2})^k.$$

It in particular implies that, under the assumption  $\{\mathfrak{X}_0^N\} \xrightarrow{\mathcal{L}} u_0(x) \rho(x) dx$  in  $M_+(\bar{D})$ ,

$$\limsup_{N \rightarrow \infty} \mathbb{E}[H_N^k(T)] \leq \|u_0\| \|\rho\| k! (C_1 T^{1/2})^k.$$

A similar argument yields  $\limsup_{N \rightarrow \infty} \mathbb{E}[G_N^k(T)] < \infty$  for any positive integer  $k$ . Hence, by interpolation, we have  $\limsup_{N \rightarrow \infty} \mathbb{E}[H_N^p(T) + G_N^p(T)] < \infty$  for all  $p \geq 1$ .

The uniform integrability implied by  $\limsup_{N \rightarrow \infty} \mathbb{E} \left[ \left( \sup_{t \in [0, T]} |H_N(t) - G_N(t)| \right)^p \right] < \infty$ , together with the convergence  $\sup_{t \in [0, T]} |H_N(t) - G_N(t)| \rightarrow 0$  in

probability, guarantee (see, e.g. Theorem 5.2 in [11, Chapter 4]) that the lemma is true. □

3.2. Estimates for evolution semigroups  $Q_{(s,t)}^N$  and  $Q_{(s,t)}$

Recall the definition of  $Q_{(s,t)}$  and  $U_{(t,s)}$  in (1.5) and (1.6), respectively. For any fixed  $t > 0$  and  $\phi \in C(\overline{D})$ ,  $v(s, x) := Q_{(s,t)}\phi(x)$  is the unique element in  $C([0, t] \times \overline{D})$  satisfying the integral equation

$$v(s, x) = P_{t-s}\phi(x) - \frac{1}{2} \int_0^{t-s} \int_{\partial D} p(\theta, x, y) q(s + \theta, y) v(s + \theta, y) \rho(y) d\sigma(y) d\theta; \tag{3.2}$$

see [5, Proposition 4.1]. We call  $v$  the **probabilistic solution** of the backward equation

$$\begin{cases} \frac{\partial v}{\partial s} = -\mathcal{A}v & \text{on } (0, t) \times D \\ \frac{\partial v}{\partial \vec{\nu}} = \frac{q v}{\rho} & \text{on } (0, t) \times \partial D \\ v(t) = \phi & \text{on } D. \end{cases} \tag{3.3}$$

Analogous to the definition of  $Q_{(s,t)}$  and  $U_{(t,s)}$ , we define

$$\begin{aligned} Q_{s,t}^N \phi(x) &:= \mathbb{E} \left[ \phi(X_t) \exp \left( - \int_s^t q_N(r, X_r) dr \right) \middle| X_s = x \right] \\ &= \mathbb{E} \left[ \phi(X_{t-s}) \exp \left( - \int_0^{t-s} q_N(s+r, X_r) dr \right) \middle| X_0 = x \right] \end{aligned} \tag{3.4}$$

and

$$U_{(t,s)}^N \mu(\phi) := \mu(Q_{s,t}^N \phi) \tag{3.5}$$

for  $\alpha > 0$ ,  $\mu \in \mathcal{H}_{-\alpha}$  and  $\phi \in L^2(D)$  whenever it is well defined (e.g.  $Q_{s,t}^N \phi \in \mathcal{H}_\alpha$ ). Then  $v_N(s, x) := Q_{(s,t)}^N \phi(x)$  is the unique element in  $C([0, t] \times \overline{D})$  satisfying the integral equation

$$v_N(s, x) = P_{t-s}\phi(x) - \frac{1}{2} \int_0^{t-s} P_\theta (q_N(s + \theta)v_N(s + \theta))(x) d\theta, \quad 0 \leq s \leq t, \tag{3.6}$$

provided that  $\phi \in C(\overline{D})$ . Here we recall that  $\{P_t; t \geq 0\}$  is the transition semigroup of  $X$  in  $L^2(D, \rho(x)dx)$ . As before,  $v_N$  is called the **probabilistic solution** of the backward

equation

$$\begin{cases} \frac{\partial v_N}{\partial s} = -\frac{1}{2}\Delta v_N + q_N v_N & \text{on } (0, t) \times D \\ \frac{\partial v_N}{\partial \bar{v}} = 0 & \text{on } (0, t) \times \partial D \\ v_N(t) = \phi & \text{on } D \end{cases} \tag{3.7}$$

**Remark 3.4.** It can be shown (cf. [7]), using the Markov property of the reflected diffusion  $X$ , that each  $Z = Z^i$  (described in Remark 1.3) is a time-inhomogeneous Markov process on  $\bar{D} \cup \{\Delta^{(i)}\}$  with  $(Q_{s,t}^N)_{s \leq t}$  being its transition operator:  $Q_{s,t}^N f(x) = \mathbb{E}[f(Z_t)|Z_s = x]$ , with the convention that  $f(\Delta) = 0$ . Besides, (3.7) is the Kolmogorov’s backward equation for  $Z$  and (3.4) is the probabilistic representation of the solution to (3.7). □

The following uniform convergence and uniform bound are useful in many places of this paper.

**Lemma 3.5.** For all  $\phi \in C(\bar{D})$  and  $0 \leq s \leq t$ , we have

$$\lim_{N \rightarrow \infty} Q_{s,t}^N \phi = Q_{s,t} \phi \quad \text{uniformly on } \bar{D} \quad \text{and} \tag{3.8}$$

$$\sup_N |Q_{s,t}^N \phi(x)| \vee |Q_{s,t} \phi(x)| \leq P_{t-s} |\phi|(x) \leq \|\phi\| \quad \text{for } x \in \bar{D}. \tag{3.9}$$

**Proof.** Estimate (3.9) follows immediately from (3.4), (1.5) and the non-negativity of  $q$ . For (3.8), note that

$$\begin{aligned} & \left| Q_{s,t}^N \phi(x) - Q_{s,t} \phi(x) \right| = \left| \mathbb{E}_x \left[ \phi(X_{t-s}) \left( e^{-\int_0^{t-s} q_N(s+r, X_r) dr} - e^{-\int_0^{t-s} q(s+r, X_r) dL_r} \right) \right] \right| \\ & \leq \|\phi\| \mathbb{E}_x \left[ \left| \int_0^{t-s} q_N(s+r, X_r) dr - \int_0^{t-s} q(s+r, X_r) dL_r \right|^2 \right] \\ & = 2 \|\phi\| \int_{r_1=0}^{t-s} \int_{r_2=r_1}^{t-s} \left( \int_D \int_D q_N(s+r_1, z_1) q_N(s+r_2, z_2) p(r_1, x, z_1) p(r_2-r_1, z_1, z_2) \right. \\ & \quad \times \rho(z_1) \rho(z_2) dz_1 dz_2 \\ & \quad - 2 \int_D \int_{\partial D} q_N(s+r_1, z_1) q(s+r_2, z_2) p(r_1, x, z_1) p(r_2-r_1, z_1, z_2) \rho(z_1) \rho(z_2) dz_1 d\sigma(z_2) \\ & \quad + \int_{\partial D} \int_{\partial D} q(s+r_1, z_1) q(s+r_2, z_2) p(r_1, x, z_1) p(r_2-r_1, z_1, z_2) \\ & \quad \left. \times \rho(z_1) \rho(z_2) d\sigma(z_1) d\sigma(z_2) \right) dr_1 dr_2, \end{aligned}$$

which converges to zero uniformly for  $x \in \bar{D}$  by Lemma 3.1.  $\square$

**Remark 3.6.** While the non-negativity of  $q$  easily implies that  $Q$  has the contraction property (3.9), we may lose this property for  $\mathbf{U}$  because intuitively the killing effect induces a jump in the system and hence can increase the fluctuation.  $\square$

The following gradient convergence is the cornerstone in Step 5 of the proof the main theorem. Its proof is based on the inequality  $\mathcal{E}(P_t f) \leq (2e t)^{-1} \|f\|_\rho^2$  (see the Appendix of [7]).

**Lemma 3.7.** For any  $0 \leq s \leq t$  and  $\phi \in C(\bar{D})$ , we have

$$\lim_{N \rightarrow \infty} \mathcal{E}\left(Q_{(s,t)}^N \phi - Q_{(s,t)} \phi\right) = 0, \tag{3.10}$$

where  $\mathcal{E}$  is the Dirichlet form of the  $(\mathcal{A}, \rho)$ -reflected diffusion defined in (1.1) and  $\mathcal{E}(u) := \mathcal{E}(u, u)$ .

**Proof.** From (3.2) and (3.6), we have

$$\begin{aligned} & Q_{(s,t)}^N \phi(x) - Q_{(s,t)} \phi(x) \\ &= \int_0^{t-s} \int_{\partial D} p(\theta, x, y) q(s + \theta, y) Q_{(s+\theta,t)} \phi(y) \rho(y) d\sigma(y) - P_\theta(q_N(s + \theta) Q_{(s+\theta,t)}^N \phi)(x) d\theta \\ &= \int_0^{t-s} P_\theta \left( q(s + \theta) Q_{(s+\theta,t)} \phi \sigma - q_N(s + \theta) Q_{(s+\theta,t)}^N \phi \right) (x) d\theta \\ &= \int_0^{t-s} P_\theta \left( h_N^{(s,t)}(\theta) \right) (x) d\theta, \end{aligned}$$

where  $h_N^{(s,t)}(\theta)$  is the signed Borel measure  $q(s + \theta, y) Q_{(s+\theta,t)} \phi(y) \rho(y) \sigma(dy) - q_N(s + \theta, y) Q_{(s+\theta,t)}^N \phi(y) \rho(y) dy$  and  $P_\theta \mu(x) := \int_{\bar{D}} p(\theta, x, y) \mu(dy)$  for any measure  $\mu$  on  $\bar{D}$ .

On the other hand, by spectral decomposition,  $\mathcal{E}(P_t f) \leq (2e t)^{-1} \|f\|_\rho^2$  (see the Appendix of [7]), where  $\|\cdot\|_\rho$  is the  $L^2(D, \rho(x)dx)$ -norm. Hence

$$\begin{aligned} \sqrt{\mathcal{E}\left(Q_{(s,t)}^N \phi(x) - Q_{(s,t)} \phi(x)\right)} &= \sqrt{\mathcal{E}\left(\int_0^{t-s} P_\theta \left( h_N^{(s,t)}(\theta) \right) (x) d\theta\right)} \\ &\leq \int_0^{t-s} \sqrt{\mathcal{E}\left(P_\theta \left( h_N^{(s,t)}(\theta) \right)\right)} d\theta \end{aligned}$$



$$\begin{aligned}
 &= \int_0^{t-s} \sqrt{\mathcal{E}\left(P_{\theta/2} P_{\theta/2} \left(h_N^{(s,t)}(\theta)\right)\right)} d\theta \\
 &\leq \int_0^{t-s} \sqrt{\frac{1}{e\theta} \left\|P_{\theta/2} \left(h_N^{(s,t)}(\theta)\right)\right\|_\rho^2} d\theta. \tag{3.11}
 \end{aligned}$$

We now show that the last quantity in (3.11) converges to zero as  $N \rightarrow \infty$ . Note that for each  $\theta \in (0, t - s)$ , the semigroup property yields

$$\left\|P_{\theta/2} \left(h_N^{(s,t)}(\theta)\right)\right\|_\rho^2 = \int_{\overline{D}} \left(P_\theta h_N^{(s,t)}(\theta)\right)(x) h_N^{(s,t)}(\theta)(dx) \rightarrow 0 \quad \text{as } N \rightarrow \infty$$

by Lemma 3.1 and the uniform convergence (3.8). By the uniform bounds (2.2) and (3.9), for  $N$  large enough which depends only on  $D$  (hence independent of  $\theta$ ), we have  $\left\|P_\theta h_N^{(s,t)}(\theta)\right\| \leq \|q\| \|\phi\| \frac{C(d,D)}{\sqrt{\theta}}$  and

$$\begin{aligned}
 \left|P_\theta h_N^{(s,t)}(\theta)\right|(\overline{D}) &= \int_{\partial D} q(s + \theta, y) Q_{(s+\theta,t)} \phi(y) \rho(y) \sigma(dy) \\
 &\quad + \int_D q_N(s + \theta, y) Q_{(s+\theta,t)}^N \phi(y) \rho(y) dy \\
 &\leq C(d, D) \|q\| \|\phi\|.
 \end{aligned}$$

Hence the last quantity in (3.11) converges to zero as  $N \rightarrow \infty$  by the Lebesgue dominated convergence theorem and the fact that

$$\left\|P_{\theta/2} \mu\right\|_\rho^2 = \int_{\overline{D}} P_\theta \mu(x) \mu(dx) \leq \|P_\theta \mu\| \cdot |\mu|(\overline{D}),$$

where  $|\mu|$  is the total variation measure of the signed measure  $\mu$ .  $\square$

Next, we explore the continuity in time for both  $Q_{s,t}$  and  $Q_{s,t}^N$ .

**Lemma 3.8.** *There exists a constant  $c \in (0, \infty)$  such that for any  $0 \leq s \leq t \leq T$  and  $k \geq 1$ ,*

$$\sup_{r \in [0,s]} \left\|Q_{(r,t)} \phi_k - Q_{(r,s)} \phi_k\right\| \leq c \|\phi_k\| \left(\lambda_k(t - s) + C \|q\| (t - s)^{1/2}\right),$$

where  $C = C(d, D, T)$  is the same constant in (2.3). Furthermore, there exists  $N_0 = N_0(D)$  such that for  $N \geq N_0$ , the above inequality holds with  $\{Q_{s,t}^N\}$  in replace of  $\{Q_{s,t}\}$ .

**Proof.** The proof will follow from a Gronwall type argument and the evolution property of the operators  $\{Q_{(s,t)}\}_{s \leq t}$ . By (3.2), for any  $0 \leq r \leq s \leq t$  and  $k$ , we have

$$\begin{aligned} & |Q_{(r,t)}\phi_k(x) - Q_{(r,s)}\phi_k(x)| \\ & \leq |e^{-\lambda_k(t-r)}\phi_k(x) - e^{-\lambda_k(s-r)}\phi_k(x)| \\ & \quad + \frac{1}{2} \left| \int_{s-r}^{t-r} \int_{\partial D} p(\theta, x, y)q(r + \theta, y)Q_{(r+\theta,t)}\phi_k(y) \rho(y) d\sigma(y) d\theta \right| \\ & \quad + \frac{1}{2} \left| \int_0^{s-r} \int_{\partial D} p(\theta, x, y)q(r + \theta, y) (Q_{(r+\theta,t)}\phi_k - Q_{(r+\theta,s)}\phi_k)(y) \rho(y) d\sigma(y) d\theta \right|. \end{aligned}$$

Now we fix  $k$ , fix  $0 \leq s \leq t$  and define  $f(r) := \|Q_{(r,t)}\phi_k - Q_{(r,s)}\phi_k\|$  for  $r \in [0, s]$ . Then the above estimate, together with (2.3) and (3.9), implies that

$$f(r) \leq A + B \int_0^{s-r} \frac{f(r + \theta)}{\sqrt{\theta}} d\theta \quad \text{for } r \in [0, s], \tag{3.12}$$

where  $A = \lambda_k \|\phi_k\|(t - s) + \|q\| C(d, D, T) \|\phi_k\|(t - s)^{1/2}$  and  $B = \frac{1}{2} C(d, D, T) \|q\|$ .

Rewriting (3.12) as  $f(r) \leq A + B \int_r^s \frac{f(w)}{\sqrt{w-r}} dw$  and keep iterating yields

$$\begin{aligned} f(r) & \leq A + AB \int_{w_1=r}^s \frac{1}{\sqrt{w_1 - r}} + AB^2 \int_{w_1=r}^s \int_{w_2=w_1}^s \frac{1}{\sqrt{(w_1 - r)(w_2 - w_1)}} \\ & \quad + AB^3 \int_{w_1=r}^s \int_{w_2=w_1}^s \int_{w_3=w_2}^s \frac{1}{\sqrt{(w_1 - r)(w_2 - w_1)(w_3 - w_2)}} + \dots \\ & = A \sum_{k=0}^{\infty} B^k a_k (s - r)^{k/2}, \text{ where } a_k = \frac{\pi^{k/2}}{\Gamma((k + 2)/2)} \text{ by Lemma A.4 in Appendix A} \\ & \leq \frac{c}{2} A \sum_{k=0}^{\infty} B^k (s - r)^{k/2} \quad \text{for some absolute constant } c > 0 \\ & \leq cA \quad \text{if } |B\sqrt{s - r}| \leq 1/2 \end{aligned}$$

Note that when  $B > 0$ ,  $|B\sqrt{s - r}| \leq 1/2$  holds if and only if  $s - \frac{1}{4B^2} \leq s \leq s + \frac{1}{4B^2}$ . (The case  $B = 0$  is trivial since then  $q = 0$ .) When  $0 \leq r < s - 1/(4B^2)$ , by the evolution property and the contraction property (3.2), we have

$$\begin{aligned} \|Q_{(r,t)}\phi_k - Q_{(r,s)}\phi_k\| & = \|Q_{(r, s-1/(4B^2))} (Q_{(s-1/(4B^2), t)}\phi_k - Q_{(s-1/(4B^2), s)}\phi_k)\| \\ & \leq \|Q_{(s-1/(4B^2), t)}\phi_k - Q_{(s-1/(4B^2), s)}\phi_k\| \leq cA \end{aligned}$$

The above arguments clearly hold with  $\{Q_{s,t}^N\}$  in replace of  $\{Q_{s,t}\}$ , if we use (2.2) instead of (2.3). This completes the proof of the lemma.  $\square$

The next lemma is a key estimate that we need to establish Theorem 4.9. Recall from (1.10) that

$$\mathcal{E}_r^{(q)}(\phi, \psi) := \langle \mathbf{a}\nabla\phi \cdot \nabla\psi, u(s) \rangle_\rho + \int_{\partial D} \phi \psi u(s) q(s) \rho d\sigma \text{ and } \mathcal{E}_r^{(q)}(\phi) := \mathcal{E}_r^{(q)}(\phi, \phi). \tag{3.13}$$

For later use, we also define

$$\mathcal{E}_s^{(q),N}(\phi, \psi) := \langle \mathbf{a}\nabla\phi \cdot \nabla\psi + q_N(s)\phi\psi, \mathfrak{X}_s^N \rangle \text{ and } \mathcal{E}_s^{(q),N}(\phi) := \mathcal{E}_s^{(q),N}(\phi, \phi). \tag{3.14}$$

**Lemma 3.9.** *For all integers  $k \geq 1$  and  $0 \leq s \leq t \leq T$ , we have*

$$\int_s^t \mathcal{E}_r^{(q)}(Q_{(r,t)}\phi_k) dr \leq C \|u_0\| (1 \vee \|q\|)^2 (\lambda_k + \|\phi_k\|^2) (t - s), \tag{3.15}$$

$$\int_0^s \mathcal{E}_r^{(q)}(Q_{(r,t)}\phi_k - Q_{(r,s)}\phi_k) dr \leq C \|u_0\| (1 \vee \|q\|)^4 (\lambda_k^2 + \|\phi_k\|^2 + \|\phi_k\|^2 \lambda_k^2) (t - s), \tag{3.16}$$

where  $C = C(d, D, T) > 0$  is a constant. Moreover, these two inequalities remain valid if we replace  $Q_{r,t}$  by  $Q_{r,t}^N$  and  $\mathcal{E}_r^{(q)}$  by  $\mathcal{E}_r^{(q),N}$  at the same time.

**Proof.** For the first inequality, note that

$$0 \leq \mathcal{E}_r^{(q)}(Q_{(r,t)}\phi_k) \leq \|u_0\| \left( \mathcal{E}(Q_{(r,t)}\phi_k) + \sigma(\partial D) \|q\| \|\rho\| \|\phi_k^2\| \right). \tag{3.17}$$

Moreover, by the integral equation (3.2), we have

$$\begin{aligned} \mathcal{E}(Q_{(r,t)}\phi_k) &\leq 2\mathcal{E}(P_{t-r}\phi_k) + 2\mathcal{E}\left(\frac{1}{2} \int_0^{t-r} P_\theta \left[ H^{(r,t)}(\theta) \right] (x) d\theta\right) \\ &= 2\lambda_k e^{-2(t-r)\lambda_k} + \mathcal{E}\left(\int_0^{t-r} P_\theta \left[ H^{(r,t)}(\theta) \right] d\theta\right), \end{aligned} \tag{3.18}$$

where  $H^{(r,t)}(\theta)$  is the signed Borel measure  $q(r + \theta, y)Q_{(r+\theta,t)}\phi_k(y) \rho(y) \sigma(dy)$  and  $P_\theta\mu(x) := \int_{\overline{D}} p(t, x, y) \mu(dy)$  for any measure  $\mu$  on  $\overline{D}$ .

By the same argument as that in the proof of Lemma 3.7, we have

$$\begin{aligned}
 \mathcal{E} \left( \int_0^{t-r} P_\theta [H^{(r,t)}(\theta)] d\theta \right) &\leq \left( \int_0^{t-r} \sqrt{\frac{1}{e\theta} \|P_{\theta/2} H^{(r,t)}(\theta)\|_\rho^2} d\theta \right)^2 \\
 &\leq \left( \int_0^{t-r} \sqrt{\frac{1}{e\theta} \|P_\theta H^{(r,t)}(\theta)\|_\infty |H^{(r,t)}(\theta)|(\overline{D})} d\theta \right)^2 \\
 &\leq \left( \int_0^{t-r} C(d, D, T) \|q\| \|\phi_k\| \theta^{-3/4} d\theta \right)^2 \\
 &\leq C(d, D, T) \|q\|^2 \|\phi_k\|^2 (t-r)^{1/2}.
 \end{aligned} \tag{3.19}$$

Now we put (3.19) into (3.18) and then put the result into (3.17) to obtain

$$\mathcal{E}_r^{(q)}(Q_{(r,t)}\phi_k) \leq \|u_0\| \left( 2\lambda_k e^{-2(t-r)\lambda_k} + C(d, D, T) \left( \|q\|^2 \|\phi_k\|^2 (t-r)^{1/2} + \|q\| \|\phi_k^2\| \right) \right). \tag{3.20}$$

By integration, we obtain

$$\int_s^t \mathcal{E}_r^{(q)}(Q_{(r,t)}\phi_k) dr \leq C(d, D, T) \|u_0\| \left( \|\phi_k\|^2 \|q\|^2 (t-s)^{3/2} + (\lambda_k + \|\phi_k^2\| \|q\|)(t-s) \right)$$

which implies (3.15).

The second inequality in the lemma can be dealt with in a similar way. More precisely, we have as in (3.17),

$$\begin{aligned}
 0 &\leq \mathcal{E}_r^{(q)}(Q_{(r,t)}\phi_k - Q_{(r,s)}\phi_k) \\
 &\leq \|u_0\| \left( \mathcal{E}(Q_{(r,t)}\phi_k - Q_{(r,s)}\phi_k) + \sigma(\partial D) \|q\| \|\rho\| \|Q_{(r,t)}\phi_k - Q_{(r,s)}\phi_k\|^2 \right)
 \end{aligned} \tag{3.21}$$

and

$$\begin{aligned}
 &\mathcal{E}(Q_{(r,t)}\phi_k - Q_{(r,s)}\phi_k) \\
 &\leq 2 \left( e^{-(t-r)\lambda_k} - e^{-(s-r)\lambda_k} \right)^2 \mathcal{E}(\phi_k) \\
 &\quad + 2 \mathcal{E} \left( \int_{s-r}^{t-r} \int_{\partial D} p(\theta, x, y) q(r + \theta, y) Q_{(r+\theta,t)}\phi_k(y) d\sigma(y) d\theta \right) \\
 &\quad + 2 \mathcal{E} \left( \int_0^{s-r} \int_{\partial D} p(\theta, x, y) q(r + \theta, y) (Q_{(r+\theta,t)}\phi_k - Q_{(r+\theta,s)}\phi_k)(y) d\sigma(y) d\theta \right)
 \end{aligned}$$

$$\begin{aligned}
 &\leq 2 \left( e^{-(t-r)\lambda_k} - e^{-(s-r)\lambda_k} \right)^2 \lambda_k \\
 &\quad + C(d, D, T) \|q\|^2 \|\phi_k\|^2 (t-s) \left( \frac{1}{\sqrt{s-r}} - \frac{1}{\sqrt{t-r}} \right) \\
 &\quad + C(d, D, T) \|q\|^2 \left( \sup_{r \in [0, s-\theta]} \|Q_{(r+\theta, t)}\phi_k - Q_{(r+\theta, s)}\phi_k\| \right)^2 (s-r)^{1/2} \\
 &\leq 2 \left( e^{-(t-r)\lambda_k} - e^{-(s-r)\lambda_k} \right)^2 \lambda_k \\
 &\quad + C(d, D, T) \|q\|^2 \|\phi_k\|^2 (t-s) \left[ \left( \frac{1}{\sqrt{s-r}} - \frac{1}{\sqrt{t-r}} \right) + (\lambda_k^2 + \|q\|^2)(s-r)^{1/2} \right].
 \end{aligned} \tag{3.22}$$

In the second last inequality, we have applied the same argument that we used to obtain (3.19). In the last inequality, we have used Lemma 3.8.

Now we put (3.22) into (3.21) and then apply Lemma 3.8 to obtain

$$\begin{aligned}
 &\int_0^s \mathcal{E}_r^{(q)}(Q_{(r,t)}\phi_k - Q_{(r,s)}\phi_k) dr \\
 &\leq \|u_0\| \left( (1 - e^{-(t-s)\lambda_k})^2(1 - e^{-2s\lambda_k}) + C(d, D, T)\|q\|^2 \|\phi_k\|^2 (t-s)^{3/2} \right. \\
 &\quad \left. + C(d, D, T)\|q\|^2 \left( \sup_{r \in [0, s]} \|Q_{(r+\theta, t)}\phi_k - Q_{(r+\theta, s)}\phi_k\| \right)^2 s^{3/2} \right. \\
 &\quad \left. + C(d, D, T)\|q\| \left( \sup_{r \in [0, s]} \|Q_{(r+\theta, t)}\phi_k - Q_{(r+\theta, s)}\phi_k\| \right)^2 s \right) \\
 &\leq \|u_0\| \lambda_k^2 (t-s)^2 (1 \wedge 2s\lambda_k) \\
 &\quad + C(d, D, T) \|u_0\| \|\phi_k\|^2 \left( \|q\|^2 (t-s)^{3/2} + (\|q\|^2 + 1) (\lambda_k^2 (t-s)^2 + \|q\|^2 (t-s)) \right) \\
 &\leq C(d, D, T) \|u_0\| \left( \lambda_k^2 (t-s)^2 + \|\phi_k\|^2 \|q\|^2 \lambda_k^2 (t-s)^2 + \|\phi_k\|^2 (\|q\|^2 + \|q\|^4)(t-s) \right).
 \end{aligned}$$

This implies (3.16).

Using (2.2) instead of (2.3), we see that the above arguments remain valid if we replace  $Q_{r,t}$  by  $Q_{r,t}^N$  and  $\mathcal{E}_r^{(q)}$  by  $\mathcal{E}_r^{(q),N}$ . This completes the proof of the lemma.  $\square$

**Remark 3.10.** From the proof above, there exists  $N_0 = N_0(D)$  such that, for  $0 \leq r \leq t \leq T$  and  $N \geq N_0$ , inequalities (3.20) and (3.22) remain valid if we replace  $Q_{r,t}$  by  $Q_{r,t}^N$  and  $\mathcal{E}_r^{(q)}$  by  $\mathcal{E}_r^{(q),N}$ .  $\square$

### 3.3. Martingales

We need the following result from [5, Lemma 6.1]. Note that it holds for every  $x \in \overline{D}$ .

**Lemma 3.11.** *Suppose  $X = \{X_t, t \geq 0; \mathbb{P}_x, x \in \overline{D}\}$  is an  $(\mathcal{A}, \rho)$ -reflected diffusion in a bounded Lipschitz domain  $D$  and  $f$  is in the domain of the Feller generator  $\text{Dom}^{\text{Feller}}(\mathcal{A})$ . Then we have*

$$M(t) := f(X_t) - f(X_0) - \int_0^t \mathcal{A}f(X_s) ds$$

is a continuous  $\mathcal{F}_t^X$ -martingale with quadratic variation  $\langle M \rangle_t = \int_0^t \mathbf{a} \nabla f \cdot \nabla f(X_s) ds$  under  $\mathbb{P}^x$  for any  $x \in \overline{D}$ . Moreover, if  $X_1$  and  $X_2$  are independent  $(\mathcal{A}, \rho)$ -reflected diffusion in  $D$  and  $M_i$  is the above  $M$  with  $X$  replaced by  $X_i$ , then the cross variation  $\langle M_1, M_2 \rangle_t = 0$ .

From Lemma 3.11, we obtain the following key martingales that we need for the study of  $\mathfrak{X}^N$ . The proof is the same as that for [5, Corollary 6.4] so it is omitted here.

**Lemma 3.12.** *Fix any positive integer  $N$ . For any  $\phi \in \text{Dom}^{\text{Feller}}(\mathcal{A})$ , we have under  $\mathbb{P}^\mu$  for any  $\mu \in E_N$ ,*

$$M_t^\phi := \langle \phi, \mathfrak{X}_t^N \rangle - \langle \phi, \mathfrak{X}_0^N \rangle - \int_0^t \langle \mathcal{A}\phi - q_N(s)\phi, \mathfrak{X}_s^N \rangle ds \quad \text{and} \tag{3.23}$$

$$N_t^\phi := \langle \phi, \mathfrak{X}_t^N \rangle^2 - \langle \phi, \mathfrak{X}_0^N \rangle^2 - \int_0^t \frac{1}{N} \langle \mathbf{a} \nabla \phi \cdot \nabla \phi, \mathfrak{X}_s^N \rangle + 2 \langle \phi, \mathfrak{X}_s \rangle \langle \mathcal{A}\phi, \mathfrak{X}_s^N \rangle - 2 \langle q_N \phi, \mathfrak{X}_s^N \rangle \langle \phi, \mathfrak{X}_s^N \rangle + \frac{1}{N} \langle q_N \phi^2, \mathfrak{X}_s^N \rangle ds \tag{3.24}$$

are r.c.l.l.  $\mathcal{F}_t^{\mathfrak{X}^N}$ -martingales under  $\mathbb{P}^\mu$  for any  $\mu \in E_N$ . Moreover,  $M_t^\phi$  has predictable quadratic variation

$$\langle M^\phi \rangle_t = \frac{1}{N} \int_0^t \langle \mathbf{a} \nabla \phi \cdot \nabla \phi + q_N(s)\phi^2, \mathfrak{X}_s^N \rangle ds. \tag{3.25}$$

From (3.25), (2.2) and Lemma 3.1, we have for all  $T > 0$ ,

$$\mathbb{E}^\mu[(M_t^\phi)^2] \leq \frac{1}{N} \left( 8(\|\phi\|^2 + \|\mathcal{A}\phi\|^2 t^2) + \|\phi^2 q\| C(d, D, T) t^{1/2} \right) \quad \text{for } t \in [0, T]. \tag{3.26}$$

#### 4. Non-equilibrium fluctuations

In this section, we present the proof of Theorem 1.5, the main result of this paper. Throughout this section, Assumptions 1.1 and 1.2 are in force. The initial distribu-

tions of the particles are assumed to be i.i.d with distribution  $u_0(x)\rho(x)dx$  for some  $u_0 \in C(\overline{D})$ .

#### 4.1. Langevin equation

This subsection represents Step 1 towards the proof of [Theorem 1.5](#) mentioned at the end of the Introduction. Recall that  $\mathcal{Y}_t^N$  is the fluctuation process defined by (1.4) and  $(\mathcal{H}_\alpha, |\cdot|_\alpha)$  is the Hilbert space defined in (2.4). We first answer question (i) in the introduction of this paper.

**Lemma 4.1.** *Whenever  $\alpha > d/2$ , we have  $\mathcal{Y}_t^N \in \mathcal{H}_{-\alpha}$  for  $t > 0$  and  $N \geq 1$ .*

**Proof.** Since our system is an i.i.d. sequence of sub-processes  $Z_i^{(N)}$  (see [Remark 1.3](#)), we easily obtain

$$\mathbb{E} [\langle \mathcal{Y}_t^N, \phi \rangle^2] = \text{Var}(\phi(Z_i^{(N)})) \leq \mathbb{E} [\phi(Z_1^N(t))^2] \leq \langle P_t \phi^2, u_0 \rangle \leq \|u_0\| \langle \phi^2, 1 \rangle. \tag{4.1}$$

Hence for  $\alpha > d/2$  and  $t \geq 0$ , by (2.6) and (2.8),

$$\mathbb{E} [|\mathcal{Y}_t^N|_{-\alpha}^2] = \sum_k \mathbb{E} [\langle \mathcal{Y}_t^N, h_k^{(\alpha)} \rangle^2] \leq \|u_0\| \sum_k (1 + \lambda_k)^{-\alpha} < \infty. \tag{4.2}$$

Then  $\mathcal{Y}_t^N \in \mathcal{H}_{-\alpha}$  a.s.  $\square$

**Remark 4.2.** The condition  $\alpha > d/2$  in the above lemma is sharp since, in view of (2.8),  $\mathbb{E} [|\mathcal{Y}_t^N|_0^2] = \infty$  when  $u_0 = 1, q = 0$  and  $\alpha \leq d/2$ .  $\square$

Unlike  $\mathbf{U}_{(t,s)}$ , we can check that  $\{\mathbf{U}_{(t,s)}^N\}_{t \geq s}$  is a strongly continuous evolution system on  $\mathcal{H}_\gamma$  with generator  $\{\mathbf{A}^{(\gamma)} + \mathbf{B}_t^{(N)}\}_{t \geq 0}$ , where  $\mathbf{A}^{(\gamma)}\mu(\phi) = \mu(\mathcal{A}\phi)$  and  $\mathbf{B}_t^{(N)}\mu(\phi) = \mu(q_N(t)\phi)$ ; see [8]. Using the fact that  $\mathcal{A}\phi_\ell = -\lambda_k \phi_k$ , we have  $|\mathbf{A}^{(\gamma)}\mu|_\gamma^2 = \sum_k (1 + \lambda_k)^\gamma \lambda_k^2 \langle \mu, \phi_k \rangle^2$ , which is finite if and only if (by Weyl’s law)  $|\mu|_{\gamma+2}^2$  is finite. Hence  $\text{Dom}(\mathbf{A}^{(\gamma)}) = \mathcal{H}_{\gamma+2}$ . Since  $q_N$  is bounded for each fixed  $N$ , we have, as operators on  $\mathcal{H}_\gamma$ ,

$$\text{Dom}(\mathbf{A}^{(\gamma)} + \mathbf{B}_t^{(N)}) = \text{Dom}(\mathbf{A}^{(\gamma)}) = \mathcal{H}_{\gamma+2} \quad \text{for all } N \geq 1. \tag{4.3}$$

Moreover,

$$|\mathbf{U}_{(t,s)}^N \mu|_\gamma^2 \leq e^{(t-s)\beta_N} |\mu|_\gamma^2 \quad \text{for some } \beta_N > 0. \tag{4.4}$$

The next result says that  $\mathcal{Y}^N$  solves the **stochastic evolution equation** in  $\mathcal{H}_{-\alpha}$ ,

$$dY_t = (\mathbf{A}^{(-\alpha)} + \mathbf{B}_t^{(N)})Y_t dt + dM_t^N, \quad Y_0 = \mathcal{Y}_0^N. \tag{4.5}$$

**Theorem 4.3.** *Suppose  $\alpha > d \vee (d/2 + 1)$ . For large enough  $N$ , there exists a r.c.l.l. square-integrable  $\mathcal{H}_{-\alpha}$ -valued martingale  $M^N = (M_t^N)_{t \geq 0}$  such that  $\mathcal{Y}^N$  satisfies the following two equivalent statements:*

(i) **(Weak solution)** *For any  $\phi \in \mathcal{H}_{-\alpha+2}$  and  $t \geq s \geq 0$ , we have  $\mathbb{P}$ -a.s.*

$$\langle \mathcal{Y}_t^N, \phi \rangle_{-\alpha} = \langle \mathcal{Y}_s^N, \phi \rangle_{-\alpha} + \int_s^t \langle (\mathbf{A}^{(-\alpha)} + \mathbf{B}_r^{(N)}) \mathcal{Y}_r^N, \phi \rangle_{-\alpha} dr + \langle M_t^N - M_s^N, \phi \rangle_{-\alpha}. \tag{4.6}$$

(ii) **(Evolution solution)** *For  $t \geq s \geq 0$ , we have  $\mathbb{P}$ -a.s.*

$$\mathcal{Y}_t^N = \mathbf{U}_{(t,s)}^N \mathcal{Y}_s^N + \int_s^t \mathbf{U}_{(t,r)}^N dM_r^N \quad \text{in } \mathcal{H}_{-\alpha}. \tag{4.7}$$

Moreover,  $M^N$  has bounded jumps and, for every  $\phi \in \mathcal{H}_\alpha$ ,  $M^N(\phi)$  is a real-valued square-integrable martingale with  $M_t^N(\phi) - M_{t-}^N(\phi) = \langle \mathfrak{X}_t^N - \mathfrak{X}_{t-}^N, \phi \rangle$  and predictable quadratic variation

$$\langle M^N(\phi) \rangle_t = \int_0^t \langle \mathbf{a} \nabla \phi \cdot \nabla \phi + q_N(s) \phi^2, \mathfrak{X}_s^N \rangle ds. \tag{4.8}$$

**Remark 4.4.** Here  $\int_0^t \mathbf{U}_{(t,s)}^N dM_s^N$  is the stochastic integral of the operator-valued function  $s \mapsto \mathbf{U}_{(t,s)}^N$  with respect to  $M^N$  on  $[0, t]$ . Its construction and its basic properties can be found in the monograph [23] of M. Metivier and J. Pellaumail (see also the book by P. Protter [25] for a more recent and comprehensive treatment for stochastic integration which used the same approach). Be aware that  $t \mapsto \int_0^t \mathbf{U}_{(t,s)}^N dM_s^N$  is *not* a martingale. However, since  $M^N$  has a r.c.l.l. version and by (4.4), we have  $\int_0^t \mathbf{U}_{(t,s)}^N dM_s^N$  has a r.c.l.l. version by the submartingale type inequality of Kotelenetz (cf. [19]).  $\square$

**Proof.** (i) and (ii) assert that  $\mathcal{Y}^N$  is a **weak solution** and an **evolution solution** of (4.5), respectively. Since  $\text{Dom}(\mathbf{A}^{(-\alpha)}) = \mathcal{H}_{-\alpha+2}$  is dense in  $\mathcal{H}_{-\alpha}$ , these two notions of solutions are equivalent by variation of constant (see Section 2.1.2 of [13]). So it suffices to prove (i).

By Lemma 3.12, for every  $\phi \in \text{Dom}^{\text{Feller}}(\mathcal{A})$ ,

$$\langle \mathcal{Y}_t^N, \phi \rangle = \langle \mathcal{Y}_0^N, \phi \rangle + \int_0^t \langle \mathcal{Y}_s^N, \mathcal{A}\phi - q_N(s)\phi \rangle ds + M_t^N(\phi), \tag{4.9}$$

where  $M_t^N(\phi)$  is a real valued  $\mathcal{F}_t^{\mathfrak{X}^N}$ -martingale with quadratic variation given by (4.8).



Note that in view of (2.9), each eigenfunction  $\phi_k$  is bounded and continuous on  $\bar{D}$  and hence is in the Feller generator of  $\mathcal{A}$ . By Doob’s inequality, (2.6), (4.8) and the fact that  $\mathbb{E}\langle \phi, \mathfrak{X}_s^N \rangle \leq \langle P_s|\phi|, u_0 \rangle$ , we have

$$\begin{aligned} & \sum_k \mathbb{E} \left[ \sup_{[0,T]} \left( M_t^N(h_k^{(\alpha)}) \right)^2 \right] \\ & \leq C(T) \sum_k \int_0^T \mathbb{E} \left[ \langle \mathbf{a} \nabla h_k^{(\alpha)} \cdot \nabla h_k^{(\alpha)} + q_N(s) (h_k^{(\alpha)})^2, \mathfrak{X}_s^N \rangle \right] ds \\ & = C(T) \sum_k (1 + \lambda_k)^{-\alpha} \int_0^T \langle \mathbf{a} \nabla \phi_k \cdot \nabla \phi_k + q_N(s) \phi_k^2, P_s u_0 \rangle_\rho ds. \end{aligned}$$

Recall that  $\int_{\partial D} \phi_k(x)^2 \sigma(dx) \leq C(d, D)(\lambda_k + 1)$  by (2.9). Hence

$$\begin{aligned} & \limsup_{N \rightarrow \infty} \sum_k \mathbb{E} \left[ \sup_{[0,T]} \left( M_t^N(h_k^{(\alpha)}) \right)^2 \right] \\ & \leq C(T) \|u_0\| T \sum_k (1 + \lambda_k)^{-\alpha} \left( \mathcal{E}(\phi_k) + C(d, D)\|q\|(\lambda_k + 1) \right) \\ & = C(T) \|u_0\| T \sum_k (1 + \lambda_k)^{-\alpha} \left( \lambda_k + C(d, D)\|q\|(\lambda_k + 1) \right) \\ & \leq C(d, D, T) \|u_0\| (1 \vee \|q\|) \sum_k \frac{1}{(1 + \lambda_k)^{\alpha-1}}, \end{aligned} \tag{4.10}$$

which by (2.8) is finite if and only if  $\alpha > d/2 + 1$ . Hence for  $\alpha > d/2 + 1$ , there is  $N_0 \geq 1$  so that for every  $N \geq N_0$ ,

$$c_N := \sum_k \mathbb{E} \left[ \sup_{[0,T]} \left( M_t^N(h_k^{(\alpha)}) \right)^2 \right] < \infty. \tag{4.11}$$

For  $\phi \in \mathcal{H}_\alpha$ ,  $\phi = \sum_{k=1}^\infty a_k h_k^{(\alpha)}$ , where  $a_k = \langle \phi, h_k^{(\alpha)} \rangle_\alpha$ . Define  $M_t^N(\phi) = \sum_{k=1}^\infty a_k \times M_t^N(h_k^{(\alpha)})$ , which is well defined in view of (4.11). Moreover, by the Doob’s maximal inequality,  $M_t^N(\phi)$  is the  $L^2$  and uniform limit in  $t \in [0, T]$  of  $\sum_{k=1}^j a_k M_t^N(h_k^{(\alpha)})$ . Hence  $M^N(\phi)$  is a real-valued r.c.l.l. square-integrable martingale with

$$\mathbb{E} [(M_T^N(\phi))^2] \leq c_N \sum_{k=1}^\infty a_k^2 = c_N \|\phi\|_\alpha^2. \tag{4.12}$$

Thus  $\langle M^N, \phi \rangle := M^N(\phi)$  with  $\phi \in \mathcal{H}_\alpha$  determines a r.c.l.l. square-integrable  $\mathcal{H}_{-\alpha}$ -valued martingale  $M^N$ . On other hand,

$$\begin{aligned}
 \sup_{t \in [0, \infty)} |M_t^N - M_{t-}^N|_{-\alpha}^2 &= \sup_{t \in [0, \infty)} \sum_k (1 + \lambda_k)^{-\alpha} \left( M_t^N(\phi_k) - M_{t-}^N(\phi_k) \right)^2 \\
 &= \sup_{t \in [0, \infty)} \sum_k (1 + \lambda_k)^{-\alpha} N \left( \mathfrak{X}_t^N(\phi_k) - \mathfrak{X}_{t-}^N(\phi_k) \right)^2 \\
 &\leq \frac{1}{N} \sum_k (1 + \lambda_k)^{-\alpha} \|\phi_k\|^2 \\
 &\leq C/N \quad \text{by (2.8), (2.9) and the assumption } \alpha > d.
 \end{aligned}$$

This in particular implies that  $M_t^N$  has bounded jumps.

Finally, since  $Dom(\mathbf{A}^{(-\alpha)}) = \mathcal{H}_{-\alpha+2}$ , (4.6) follows from (4.9) provided that  $\alpha > d/2 + 1$ . This completes the proof.  $\square$

#### 4.2. Convergence of $M^N$ and tightness of $\mathcal{Y}^N$

This subsection represents Step 2 and Step 3 towards the proof of Theorem 1.5. By Prohorov’s theorem, a sequence of  $\mathcal{H}_{-\alpha}$ -processes  $\{R_N\}$  is tight in  $D([0, T], \mathcal{H}_{-\alpha})$  provided that it satisfies the two conditions below:

(1) For all  $t \in [0, T]$  and  $\varepsilon_0 > 0$ , there exists  $K > 0$  such that

$$\overline{\lim}_{N \rightarrow \infty} \mathbb{P} (|R_N(t)|_{-\alpha}^2 > K) < \varepsilon_0. \tag{4.13}$$

(2) For all  $\varepsilon_0 > 0$ , as  $\delta \rightarrow 0$  we have

$$\overline{\lim}_{N \rightarrow \infty} \mathbb{P} \left( \sup_{\substack{|t-s| < \delta \\ 0 \leq s, t \leq T}} |R_N(t) - R_N(s)|_{-\alpha}^2 > \varepsilon_0 \right) \rightarrow 0. \tag{4.14}$$

Moreover, (4.14) implies that any limit point has its law concentrates on  $C([0, T], \mathcal{H}_{-\alpha})$ . The following “weak tightness criterion” can be easily checked by using (4.13), (4.14), the Chebyshev’s inequality, the metric of  $\mathcal{H}_{-\alpha}$ .

**Lemma 4.5.** *Suppose  $\{R_N; N \geq 1\}$  is a sequence of  $\mathcal{H}_{-\alpha}$ -processes for some  $\alpha \in \mathbb{R}$  such that for any  $\varepsilon_0 > 0$ ,*

$$\overline{\lim}_{N \rightarrow \infty} \mathbb{P} \left( \sup_{t \in [0, T]} \sum_{|k| > K} \langle R_N(t), h_k^{(\alpha)} \rangle^2 > \varepsilon_0 \right) \rightarrow 0 \quad \text{as } K \rightarrow \infty. \tag{4.15}$$

*Then the tightness of  $\{R_N\}$  in  $D([0, T], \mathcal{H}_{-\alpha})$  follows from the tightness of the one-dimensional processes  $\{\langle R_N, h_k^{(\alpha)} \rangle\}_{N \geq 1}$  (for all  $k \in \mathbb{N}^d$ ).*

The following result is Step 2 towards the proof of [Theorem 1.5](#).

**Theorem 4.6.** *When  $\alpha > d \vee (d/2 + 1)$ , the square-integrable martingale  $M^N$  in [Theorem 4.3](#) converges to  $M$  in distribution in  $D([0, \infty), \mathcal{H}_{-\alpha})$  as  $N \rightarrow \infty$ , where  $M$  is the (unique in distribution) continuous  $\mathcal{H}_{-\alpha}$ -valued square-integrable Gaussian martingale with independent increments and covariance functional characterized by [\(1.8\)](#).*

**Proof.** We first prove the existence and uniqueness of  $M$ . Recall the bilinear forms  $\mathcal{E}_t^{(q)}$  defined by [\(1.10\)](#). Fix  $\alpha > d \vee (d/2 + 1)$  and define a self-adjoint operator  $A(t)$  on  $\mathcal{H}_{-\alpha}$  by

$$\langle A(t)\varphi^*, \psi^* \rangle_{-\alpha} = \int_0^t \mathcal{E}_s^{(q)}(J(\varphi^*), J(\psi^*)) ds, \tag{4.16}$$

where  $J : \mathcal{H}_{-\alpha} \rightarrow \mathcal{H}_\alpha$  denotes the Riesz representation, i.e. for  $\varphi^* \in \mathcal{H}_{-\alpha}$  and  $\psi \in \mathcal{H}_\alpha$ , we have  $\langle \varphi^*, \psi \rangle = \langle \psi, J(\varphi^*) \rangle_\alpha$ . Then  $A(t)$  is a self-adjoint compact operator on the Hilbert space  $\mathcal{H}_{-\alpha}$  of finite trace because

$$\sum_k \langle A(t)h_k^{(-\alpha)}, h_k^{(-\alpha)} \rangle_{-\alpha} = \sum_k \int_0^t \mathcal{E}_s^{(q)}(h_k^{(\alpha)}, h_k^{(\alpha)}) ds < \infty$$

by a calculation similar to [\(4.10\)](#). Moreover,  $\langle A(t)\varphi^*, \varphi^* \rangle_{-\alpha}$  is a positive-definite quadratic functional of  $\varphi^*$  for every  $t$ , and is continuous and increasing in  $t$  for every  $\varphi^*$ . Hence (cf. [\[16\]](#) for a proof using Kolmogorov’s extension theorem) there is a unique (in distribution)  $\mathcal{H}_{-\alpha}$ -valued Gaussian process  $M$  on some probability space  $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$  with independent increments, continuous sample paths, and characteristic functional

$$\tilde{\mathbb{E}} \exp(i \langle M_t, \varphi^* \rangle_{-\alpha}) = \exp\left(-\frac{1}{2} \langle A(t)\varphi^*, \varphi^* \rangle_{-\alpha}\right). \tag{4.17}$$

The tightness of  $\{M^N\}$  and continuity of any limit are implied by [Lemma 4.5](#) and [\(4.10\)](#). Hence we only need to identify any subsequential limit. Observe that  $\mathbb{P}$ -a.s. we have

$$\sup_{t \in [0, T]} \left| M_t^N(\phi) - M_{t-}^N(\phi) \right| = \sup_{t \in [0, T]} \sqrt{N} \left| \mathfrak{X}_t^N(\phi) - \mathfrak{X}_{t-}^N(\phi) \right| \leq \frac{1}{\sqrt{N}} \|\phi\| \rightarrow 0 \tag{4.18}$$

and that by [Theorem 1.4](#), the quadratic variation [\(4.8\)](#) of  $M_t^N(\phi)$  converges to the deterministic quantity [\(1.8\)](#) in probability for any  $t \geq 0$ . These two observations imply, by a standard functional central limit theorem for semi-martingales (see, e.g., [\[22\]](#)), that  $\{M^N(\phi)\}$  converges to  $M(\phi)$  in distribution in  $D([0, T], \mathbb{R})$  for any  $\phi \in \text{Dom}^{Feller}(\mathcal{A})$ . Finally, since  $\mathcal{H}_\alpha$  has a countable dense subset in  $\text{Dom}^{Feller}(\mathcal{A})$  (for example, the linear

span of eigenfunctions), and since any subsequential limit of  $M^N$  is continuous in  $t$ , we know that the subsequential limit is indeed  $M$ . The proof is now complete.  $\square$

Here is Step 3 towards the proof of [Theorem 1.5](#).

**Theorem 4.7.** *The sequence of processes  $\{\mathcal{Y}^N\}$  is tight in  $D([0, T], \mathcal{H}_{-\alpha})$  whenever  $\alpha > d \vee (d/2 + 2)$ . Moreover, any subsequential limit has a continuous version.*

**Proof.** We first verify [\(4.15\)](#) for  $Y^N$ . By [\(4.9\)](#), we have

$$\begin{aligned} & \mathbb{E} \left[ \sup_{[0, T]} \langle \mathcal{Y}_t^N, \phi \rangle^2 \right] \\ & \leq C(T) \mathbb{E} \left[ \langle \mathcal{Y}_0^N, \phi \rangle^2 + \int_0^T \langle \mathcal{Y}_s^N, \mathcal{A}\phi \rangle^2 ds + \left( \int_0^T \langle \mathcal{Y}_s^N, q_N(s)\phi \rangle ds \right)^2 + \sup_{[0, T]} (M_t^N(\phi))^2 \right]. \end{aligned}$$

Observe that we have treated the second term and the third term (which involve  $q_N$ ) in the right-hand side in a different way. This is because  $\int_0^T \mathbb{E} \langle \mathcal{Y}_s^N, q_N(s)\phi \rangle^2 ds$  tends to infinity when  $q$  and  $u_0$  are strictly positive. The first two terms in the right-hand side can be estimated using the fact  $\mathbb{E}[\langle \mathcal{Y}_s^N, \phi \rangle^2] \leq \|u_0\| \langle \phi^2, 1 \rangle$  proved in [\(4.1\)](#). The martingale term can be estimated as in [\(4.10\)](#). For the third term which involve  $q_N$ , using the fact that  $(\int_s^t f(r) dr)^2 = 2 \int_s^t \int_u^t f(u)f(v) dv du$  and [\(2.2\)](#), we can check that

$$\mathbb{E} \left[ \left( \int_s^t \langle \mathcal{Y}_r^N, q_N(r)\phi \rangle dr \right)^2 \right] \leq C(D, T) \|\phi\|^2 (t - s)^{3/2} \quad \text{for } N \geq N_0(D). \tag{4.19}$$

Combining the above calculations, we have

$$\overline{\lim}_{N \rightarrow \infty} \sum_{k > K} \mathbb{E} \left[ \sup_{[0, T]} \langle \mathcal{Y}_t^N, h_k^{(\alpha)} \rangle^2 \right] \leq C(D, T) \|u_0\| \sum_{k > K} \frac{1 + \lambda_k^2 + \|\phi_k^2\| + \lambda_k}{(1 + \lambda_k)^\alpha}$$

which, by [\(2.9\)](#) and Weyl’s law [\(2.8\)](#), tends to 0 as  $K \rightarrow \infty$ , provided that  $\alpha > d \vee (d/2 + 2)$ . We conclude by Chebyshev’s inequality that [\(4.15\)](#) for  $\mathcal{Y}^N$  (in place of  $R_N$ ) is satisfied if  $\alpha > d \vee (d/2 + 2)$ .

By [Lemma 4.5](#), it remains to verify that the one-dimensional processes  $\{\langle \mathcal{Y}^N, \phi_k \rangle; N \geq 1\}$  (for all  $k \in \mathbb{N}$ ) are tight. Since  $\mathbb{E}[\langle \mathcal{Y}_t^N, \phi \rangle^2] \leq \|u_0\| \langle \phi^2, 1 \rangle$  by [\(4.1\)](#), it is enough to show that

$$\overline{\lim}_{N \rightarrow \infty} \mathbb{P} \left( \sup_{\substack{|t-s| \leq \delta \\ 0 \leq s, t \leq T}} |\langle \mathcal{Y}_t^N, \phi_k \rangle - \langle \mathcal{Y}_s^N, \phi_k \rangle| > \varepsilon_0 \right) \rightarrow 0 \quad \text{as } \delta \rightarrow 0 \tag{4.20}$$

for any  $k \in \mathbb{N}$ . Note that (4.20) together with (4.15) for  $\mathcal{Y}^N$  imply that any subsequential limit of  $\{\mathcal{Y}^N; N \geq 1\}$  has a continuous version. Since  $\mathcal{A}\phi_k$  is uniformly bounded and  $\widehat{M}_t^N(\phi_k)$  defined by (4.9) converge in  $D([0, T], \mathbb{R})$  as  $N \rightarrow \infty$  by Theorem 4.6, it remains to show that

$$\overline{\lim}_{N \rightarrow \infty} \mathbb{P} \left( \sup_{\substack{|t-s| < \delta \\ 0 \leq s, t \leq T}} \left| \int_s^t \langle \mathcal{Y}_r^N, q_N(r)\phi_k \rangle dr \right| > \varepsilon_0 \right) \rightarrow 0 \quad \text{as } \delta \rightarrow 0. \tag{4.21}$$

For this, note that even though  $\int_0^T \mathbb{E} \langle q_N(s), \mathcal{Y}_s^N \rangle^2 ds$  tends to infinity when  $q$  and  $u_0$  are strictly positive, we have (4.19) holds. Hence we have (4.21). See, for example, Problem 4.11 in Chapter 2 of [18].  $\square$

### 4.3. Convergence of transportation part

The goal of this subsection is to prove the following result, which is Step 4 towards the proof of Theorem 1.5.

**Theorem 4.8.** *For  $\alpha > d + 2$ , as  $N \rightarrow \infty$*

$$\mathbf{U}_{(t,0)}^N \mathcal{Y}_0^N \xrightarrow{\mathcal{L}} \mathbf{U}_{(t,0)} \mathcal{Y}_0 \quad \text{in } C([0, T], \mathcal{H}_{-\alpha}). \tag{4.22}$$

Moreover,  $\mathbf{U}_{(t,0)} \mathcal{Y}_0$  has a version in  $C^\gamma([0, T], \mathcal{H}_{-\alpha})$  for any  $\gamma \in (0, 1/2)$ .

**Proof. (i) Continuity of the limit.** We first prove that  $\mathbf{U}_{(\cdot,0)} \mathcal{Y}_0$  has a version in  $C^\gamma([0, T], \mathcal{H}_{-\alpha})$  for any  $\gamma \in (0, 1/2)$ . Precisely, we will show that for  $\alpha > d + 2$  and  $n \in \mathbb{N}$ ,

$$\begin{aligned} & \mathbb{E} \left[ \left| \mathbf{U}_{(t,0)} \mathcal{Y}_0 - \mathbf{U}_{(s,0)} \mathcal{Y}_0 \right|_{-\alpha}^{2n} \right] \\ & \leq C \|u_0\|^n \left( (t-s)^{2n} + \|q\|^{2n} (t-s)^n \right) \quad \text{whenever } 0 \leq s \leq t \leq T, \end{aligned} \tag{4.23}$$

where  $C = C(n, d, D, T, \alpha) \in (0, \infty)$  is a constant independent of  $s$  and  $t$ . By Kolmogorov continuity criteria, (4.23) implies the desired Hölder continuity.

From Lemma 3.8, we have

$$\begin{aligned} \mathbb{E} \left[ \langle \mathbf{U}_{(t,0)} \mathcal{Y}_0 - \mathbf{U}_{(s,0)} \mathcal{Y}_0, \phi_k \rangle^2 \right] & \leq \|u_0 \rho\| \langle (Q_{0,t} \phi_k - Q_{0,s} \phi_k)^2, 1 \rangle \\ & \leq \|u_0\| C(d, D, T) \|\phi_k\|^2 \left( \lambda_k^2 (t-s)^2 + \|q\|^2 (t-s) \right). \end{aligned}$$

Using the Gaussian property of  $\langle \mathbf{U}_{(t,0)} \mathcal{Y}_0 - \mathbf{U}_{(s,0)} \mathcal{Y}_0, \phi \rangle$ , the above inequality and the simple fact  $(a + b)^n \leq 2^n (a^n + b^n)$ , we have

$$\begin{aligned} \mathbb{E} [\langle \mathbf{U}_{(t,0)}\mathcal{Y}_0 - \mathbf{U}_{(s,0)}\mathcal{Y}_0, \phi_k \rangle^{2n}] &= (2n - 1)!! (\mathbb{E} [\langle \mathbf{U}_{(t,0)}\mathcal{Y}_0 - \mathbf{U}_{(s,0)}\mathcal{Y}_0, \phi \rangle^2])^n \\ &\leq (2n - 1)!! 2^n C^n(d, D, T) \|u_0\|^n \|\phi_k\|^{2n} (\lambda_k^{2n}(t - s)^{2n} + \|q\|^{2n} (t - s)^n). \end{aligned}$$

Therefore, using Hölder inequality  $(\sum_i a_i b_i)^n \leq (\sum_i a_i^{n/(n-1)})^{n-1} (\sum_i b_i^n)$  for non-negative numbers  $a_i$  and  $b_i$ , we have for any  $\beta \in (0, \alpha]$ ,

$$\begin{aligned} &\mathbb{E} \left[ |\mathbf{U}_t \mathcal{Y}_0 - \mathbf{U}_s \mathcal{Y}_0|_{-\alpha}^{2n} \right] \\ &= \mathbb{E} \left[ \left( \sum_k (1 + \lambda_k)^{-\alpha} \langle \mathbf{U}_t \mathcal{Y}_0 - \mathbf{U}_s \mathcal{Y}_0, \phi_k \rangle^2 \right)^n \right] \\ &\leq \left( \sum_k (1 + \lambda_k)^{-\frac{\beta n}{n-1}} \right)^{n-1} \left( \sum_k (1 + \lambda_k)^{-(\alpha-\beta)n} \mathbb{E} [\langle \mathbf{U}_t \mathcal{Y}_0 - \mathbf{U}_s \mathcal{Y}_0, \phi_k \rangle^{2n}] \right) \\ &\leq C(n, d, D, T) \|u_0\|^n \left( \sum_k \frac{1}{(1 + \lambda_k)^{\frac{\beta n}{n-1}}} \right)^{n-1} \\ &\quad \cdot \left( (t - s)^{2n} \sum_k \frac{\|\phi_k\|^{2n} \lambda_k^{2n}}{(1 + \lambda_k)^{(\alpha-\beta)n}} + \|q\|^{2n} (t - s)^n \sum_k \frac{\|\phi_k\|^{2n}}{(1 + \lambda_k)^{(\alpha-\beta)n}} \right). \end{aligned}$$

From (2.9), it follows that (4.24) holds true once we choose  $\beta \in \left(\frac{d(n-1)}{2n}, \alpha - \frac{d}{2} - 2 - \frac{d}{2n}\right)$ . This choice of  $\beta$  is possible if and only if  $\alpha > d + 2$ . Hence the proof of (4.23) is complete.

(ii) **Tightness.** Next, we show that  $\{\mathbf{U}_{(t,0)}^N \mathcal{Y}_0^N\}$  is tight in  $C([0, T], \mathcal{H}_{-\alpha})$ . Let  $\psi = Q_{0,t}^N \phi_k - Q_{0,s}^N \phi_k$  and  $\{x_i\}_{i=1}^N$  be i.i.d. with distribution  $u_0(x)\rho(x)dx$ . Then

$$\begin{aligned} &\mathbb{E} [\langle \mathbf{U}_{(t,0)}^N \mathcal{Y}_0^N - \mathbf{U}_{(s,0)}^N \mathcal{Y}_0^N, \phi_k \rangle^4] = \mathbb{E} [\langle \mathcal{Y}_0^N, \psi \rangle^4] \\ &= N^2 \mathbb{E} \left[ \left( \frac{1}{N} \sum_{i=1}^N (\psi(x_i) - \mu_\psi) \right)^4 \right] \quad (\text{where } \mu_\psi := \langle \psi, u_0 \rangle_\rho \text{ is the mean of each term}) \\ &= \frac{1}{N} \mathbb{E}[(\psi(x_1) - \mu_\psi)^4] + \left( \mathbb{E}[(\psi(x_1) - \mu_\psi)^2] \right)^2 \\ &\leq C(d, D, T) \|u_0 \rho\|^2 \|\phi_k\|^4 \left( \lambda_k^4 (t - s)^4 + \|q\|^4 (t - s)^2 \right) \quad (\text{by Lemma 3.8}). \end{aligned}$$

Using Hölder inequality  $(\sum_i a_i b_i)^n \leq (\sum_i a_i^{n/(n-1)})^{n-1} (\sum_i b_i^n)$  as in step (i) above (with  $n = 2$  here), we obtain

$$\sup_{N>1} \mathbb{E} \left[ |\mathbf{U}_{(t,0)}^N \mathcal{Y}_0^N - \mathbf{U}_{(s,0)}^N \mathcal{Y}_0^N|_{-\alpha}^4 \right] \leq C \|u_0\|^2 ((t - s)^4 + \|q\|^4 (t - s)^2) \quad (4.24)$$

whenever  $0 \leq s \leq t \leq T$  and  $\alpha > d + 2$ , where  $C = C(\alpha, d, D, T) \in (0, \infty)$ . Inequality (4.24) implies the tightness we need, in view of the Kolmogorov–Centov tightness criteria (see [12, Theorem 3.8.8]).

(iii) **Convergence of finite dimensional distributions.** To finish the proof of [Theorem 4.8](#), it remains to show that for any  $n \in \mathbb{N}$  and  $0 \leq t_1 \leq \dots \leq t_n < \infty$ , we have

$$\left( \mathbf{U}_{(t_1,0)}^N \mathcal{Y}_0^N, \dots, \mathbf{U}_{(t_n,0)}^N \mathcal{Y}_0^N \right) \xrightarrow{\mathcal{L}} \left( \mathbf{U}_{(t_1,0)} \mathcal{Y}_0, \dots, \mathbf{U}_{(t_n,0)} \mathcal{Y}_0 \right) \quad \text{in } (\mathcal{H}_{-\alpha})^n \quad (4.25)$$

as  $N \rightarrow \infty$ , where  $(\mathcal{H}_{-\alpha})^n$  has the product topology.

For this, it suffices to show that for any  $\psi_1, \dots, \psi_n \in \mathcal{C} \subset C(\overline{D})$ ,

$$\begin{aligned} & \left( \langle \mathbf{U}_{(t_1,0)}^N \mathcal{Y}_0^N, \psi_1 \rangle, \dots, \langle \mathbf{U}_{(t_n,0)}^N \mathcal{Y}_0^N, \psi_n \rangle \right) \\ & \xrightarrow{\mathcal{L}} \left( \langle \mathbf{U}_{(t_1,0)} \mathcal{Y}_0, \psi_1 \rangle, \dots, \langle \mathbf{U}_{(t_n,0)} \mathcal{Y}_0, \psi_n \rangle \right) \quad \text{in } \mathbb{R}^n, \end{aligned} \quad (4.26)$$

where  $\mathcal{C}$  denotes the linear span of the eigenfunctions  $\{\phi_k\}$ . This is because  $\mathcal{C}$  is dense in  $\mathcal{H}_\alpha$  and the Borel  $\sigma$ -field in  $(\mathcal{H}_{-\alpha})^n$  is generated by the finite dimensional sets.

We first prove [\(4.26\)](#) when  $n = 1$ . For notational simplicity, write  $t$  and  $\psi$  for  $t_1$  and  $\psi_1$ . Note that  $\{\langle \mathbf{U}_{(t,0)}^N \mathcal{Y}_0^N, \psi \rangle\}_{N \geq 1}$  is tight in  $\mathbb{R}$  since by [\(3.9\)](#),

$$\sup_N \mathbb{E} [\langle \mathbf{U}_t^N \mathcal{Y}_0^N, \psi \rangle^2] = \sup_N \langle (Q_{0,t}^N \psi)^2, u_0 \rangle - \langle Q_{0,t}^N \psi, u_0 \rangle^2 \leq \|u_0\| \sup_N \langle (Q_{0,t}^N \psi)^2, 1 \rangle < \infty.$$

Suppose  $Z$  is a subsequential limit of  $\langle \mathbf{U}_{(t,0)}^N \mathcal{Y}_0^N, \psi \rangle$ . We claim that  $Z \stackrel{\mathcal{L}}{=} \langle \mathbf{U}_{(t,0)} \mathcal{Y}_0, \psi \rangle$ .

This is due to the following two facts:  $\langle \mathcal{Y}_0^N, Q_{0,t} \psi \rangle \xrightarrow{\mathcal{L}} \langle \mathcal{Y}_0, Q_{0,t} \psi \rangle$  (by the standard central limit theorem) and  $\lim_{N \rightarrow \infty} \mathbb{E} |\langle \mathcal{Y}_0^N, Q_{0,t} \psi \rangle - \langle \mathcal{Y}_0, Q_{0,t} \psi \rangle| = 0$ , which follows from

$$\begin{aligned} \mathbb{E} \left[ |\langle \mathcal{Y}_0^N, Q_{0,t} \psi \rangle - \langle \mathcal{Y}_0, Q_{0,t} \psi \rangle|^2 \right] &= \langle (Q_{0,t}^N \psi - Q_{0,t} \psi)^2, u_0 \rangle - \langle Q_{0,t}^N \psi - Q_{0,t} \psi, u_0 \rangle^2 \\ &\leq \|u_0\| \langle (Q_{0,t}^N \psi - Q_{0,t} \psi)^2, 1 \rangle \rightarrow 0 \quad \text{by } (3.8). \end{aligned}$$

In fact, the second fact implies that  $\langle \mathcal{Y}_0^{N'}, Q_{0,t} \psi \rangle - \langle \mathcal{Y}_0, Q_{0,t} \psi \rangle \rightarrow 0$  a.s. along some subsequence  $N'$ , and so by the Lebesgue dominated convergence theorem,  $\mathbb{E} F(\langle \mathcal{Y}_0^{N'}, Q_{0,t} \psi \rangle) - \mathbb{E} F(\langle \mathcal{Y}_0, Q_{0,t} \psi \rangle) \rightarrow 0$  for any bounded continuous function  $F$ .

The proof of [\(4.26\)](#) for general  $n \in \mathbb{N}$  is the same as that for  $n = 1$ , using the standard multidimensional central limit theorem. So we get the desired [\(4.25\)](#).

The proof of [Theorem 4.8](#) is now complete.  $\square$

#### 4.4. Convergence of stochastic integrals

Our goal in this subsection is to prove the following result, which corresponds to Step 5 towards the proof of [Theorem 1.5](#).

**Theorem 4.9.** For  $\alpha > d + 2$  and  $T > 0$ , as  $N \rightarrow \infty$

$$\int_0^t \mathbf{U}_{(t,s)}^N dM_s^N \xrightarrow{\mathcal{L}} \int_0^t \mathbf{U}_{(t,s)} dM_s \quad \text{in } D([0, T], \mathcal{H}_{-\alpha}). \tag{4.27}$$

Moreover,  $\int_0^t \mathbf{U}_{(t,s)} dM_s$  has a version in  $C^\gamma([0, T], \mathcal{H}_{-\alpha})$  for any  $\gamma \in (0, 1/2)$ .

First, we need the following lemma which is the key for establishing finite dimensional convergence. [Lemma 3.9](#) also plays a crucial role in the proof of [Theorem 4.9](#). Recall from [\(3.13\)](#) and [\(3.14\)](#) that

$$\begin{aligned} \mathcal{E}_r^{(q)}(\phi, \psi) &:= \langle \mathbf{a} \nabla \phi \cdot \nabla \psi, u(s) \rangle_\rho + \int_{\partial D} \phi \psi u(s) q(s) \rho d\sigma, \quad \mathcal{E}_r^{(q)}(\phi) := \mathcal{E}_r^{(q)}(\phi, \phi), \quad \text{and} \\ \mathcal{E}_s^{(q),N}(\phi, \psi) &:= \langle \mathbf{a} \nabla \phi \cdot \nabla \psi + q_N(s) \phi \psi, \mathfrak{X}_s^N \rangle, \quad \mathcal{E}_s^{(q),N}(\phi) := \mathcal{E}_s^{(q),N}(\phi, \phi). \end{aligned}$$

**Lemma 4.10.** For  $0 \leq a \leq b \leq T$ ,  $i = \sqrt{-1}$  and  $\phi \in C(\bar{D})$ , as  $N \rightarrow \infty$ , we have

$$\begin{aligned} \mathbb{E} \left[ \exp \left( i \left\langle \int_a^b \mathbf{U}_{(T,s)}^N dM_s^N, \phi \right\rangle \right) \middle| \mathcal{F}_a^N \right] &\text{ converges in } L^1(\mathbb{P}) \text{ to} \\ \exp \left( -\frac{1}{2} \int_a^b \mathcal{E}_s^{(q)}(Q_{(s,T)} \phi) ds \right) &= \mathbb{E} \left[ \exp \left( i \left\langle \int_a^b \mathbf{U}_{(T,s)} dM_s, \phi \right\rangle \right) \right]. \end{aligned}$$

**Proof.** (i) Fix  $T > 0$  and  $\phi \in \mathcal{H}_\alpha$ . Then

$$K_t = K_t^N := \left\langle \int_0^t \mathbf{U}_{(T,s)}^N dM_s^N, \phi \right\rangle \quad \text{is a martingale for } t \in [0, T].$$

Let  $\Delta K_r := K_r - K_{r-}$  denote the jump of  $K$  at time  $r$ . Then by [\(4.18\)](#) and [\(3.9\)](#),

$$\sup_{r \in [0, T]} |\Delta K_r| \leq \sup_{s \in [0, T]} \|Q_{s,T}^N \phi\| / \sqrt{N} \leq \|\phi\| / \sqrt{N}. \tag{4.28}$$

Moreover, by [Theorem 4.3](#), the dual predictable projection  $\langle K \rangle$  of the quadratic variation  $[K]$  of  $K$  is

$$\langle K \rangle_t = \int_0^t \mathcal{E}_s^{(q),N}(Q_{(s,T)}^N \phi) ds \quad \text{for } t \in [0, T]. \tag{4.29}$$

By a similar argument as that for  $H_N(t)$  in the proof of [Lemma 3.3](#) (using an inequality in [Remark 3.10](#)), we have  $\limsup_{N \rightarrow \infty} \mathbb{E}[\langle K \rangle_T^k] < \infty$  for every integer  $k \geq 1$ . Observe



that  $J_t := [K]_t - \langle K \rangle_t$  is a purely discontinuous martingale with jumps  $\Delta J_t := J_t - J_{t-} = (\Delta K_t)^2$ . It follows from (4.28) that

$$\begin{aligned} \mathbb{E}[J_T^2] &= \mathbb{E} \left[ \sum_{0 < s \leq T} (\Delta J_s)^2 \right] \leq \|\phi\|^2 N^{-1} \mathbb{E} \left[ \sum_{0 < s \leq T} (\Delta K_s)^2 \right] \leq \|\phi\|^2 N^{-1} \mathbb{E}[K]_T \\ &= \|\phi\|^2 N^{-1} \mathbb{E}\langle K \rangle_T. \end{aligned}$$

Hence

$$\mathbb{E} [[K]_T^2] = \mathbb{E} [\langle K \rangle_T + J_T]^2 \leq 2\mathbb{E} [\langle K \rangle_T^2 + J_T^2] \leq 2\mathbb{E}\langle K \rangle_T + 2\|\phi\|^2 N^{-1} \mathbb{E}\langle K \rangle_T, \tag{4.30}$$

which is uniformly bounded in  $N$ , by Lemma 3.9.

Let  $f(r) := e^{ir}$ ,  $g(r) := \mathcal{E}_r^{(q)}(Q_{r,T}\phi)$ , and  $g_N(r) := \mathcal{E}_r^{(q),N}(Q_{r,T}^N\phi)$ . Fix  $a \in [0, T]$ , and set  $h_N(t) := \mathbb{E} [f(K_t - K_a) | \mathcal{F}_a^N]$  and  $h(t) := \exp\left(-\frac{1}{2} \int_a^t g(r) dr\right)$ . Note that  $h(t) = 1 - \frac{1}{2} \int_a^t h(r)g(r)dr$ . We claim that

$$h_N(t) = 1 - \frac{1}{2} \int_a^t h_N(r)g(r)dr + \varepsilon_N(t) \quad \text{with} \quad \sup_{t \in [a, T]} |\varepsilon_N(t)| \rightarrow 0 \quad \text{in} \quad L^1(\mathbb{P}) \tag{4.31}$$

as  $N \rightarrow \infty$ . By Gronwall’s inequality, the above equations yield

$$|h_N(t) - h(t)| \leq \left( \sup_{t \in [a, T]} |\varepsilon_N(t)| \right) \exp\left(\frac{1}{2} \int_a^t g(r) dr\right)$$

and hence  $h_N(t) \rightarrow h(t)$  in  $L^1(\mathbb{P})$  as  $N \rightarrow \infty$ . On other hand, since  $M$  is Gaussian with independent increment,  $\langle \int_a^b \mathbf{U}_{(c,s)} dM_s, \phi \rangle$  is Gaussian with variance  $\int_a^b \mathcal{E}_s^{(q)}(Q_{(s,T)}\phi) ds$  (see Subsection A.2 in Appendix A). Thus we have  $\exp\left(-\frac{1}{2} \int_a^b \mathcal{E}_s^{(q)}(Q_{(s,T)}\phi) ds\right) = \mathbb{E} \left[ \exp\left(i \langle \int_a^b \mathbf{U}_{(T,s)} dM_s, \phi \rangle\right) \right]$ . This proves Lemma 4.10 once the claim (4.31) is verified. We now prove (4.31) in the next two steps.

(ii) By Itó’s formula (see, e.g., Theorem 36 in [25, Chapter II]),

$$\begin{aligned} f(K_t) &= 1 + \int_{0+}^t f'(K_{r-})dK_r + \frac{1}{2} \int_{0+}^t f''(K_{r-})d[K]_r \\ &\quad + \sum_{0 < r \leq t} \left( f(K_r) - f(K_{r-}) - f'(K_{r-})\Delta K_r - \frac{1}{2} f''(K_{r-})(\Delta K_r)^2 \right). \end{aligned} \tag{4.32}$$

Hence for  $t \in [a, T]$ ,

$$\begin{aligned} & \mathbb{E} [f(K_t) | \mathcal{F}_a^N] \\ &= f(K_a) + \mathbb{E} \left[ \frac{1}{2} \int_{a+}^t f''(K_{r-}) d[K]_r | \mathcal{F}_a^N \right] \\ & \quad + \mathbb{E} \left[ \sum_{0 < r \leq t} \left( f(K_r) - f(K_{r-}) - f'(K_{r-}) \Delta K_r - \frac{1}{2} f''(K_{r-}) (\Delta K_r)^2 \right) | \mathcal{F}_a^N \right] \\ &= f(K_a) - \frac{1}{2} \mathbb{E} \left[ \int_{a+}^t f(K_{r-}) g(r) dr | \mathcal{F}_a^N \right] + \varepsilon_N^{(1)}(t) + \varepsilon_N^{(2)}(t), \end{aligned}$$

where

$$\begin{aligned} \varepsilon_N^{(1)}(t) &:= \frac{1}{2} \mathbb{E} \left[ \int_{a+}^t f(K_{r-}) (g(r) - g_N(r)) dr | \mathcal{F}_a^N \right] \quad \text{and} \\ \varepsilon_N^{(2)}(t) &:= \mathbb{E} \left[ \sum_{0 < r \leq t} \left( f(K_r) - f(K_{r-}) - f'(K_{r-}) \Delta K_r - \frac{1}{2} f''(K_{r-}) (\Delta K_r)^2 \right) | \mathcal{F}_a^N \right]. \end{aligned}$$

We have used (4.29) and the fact that  $f'' = -f$  in the last equality.

Dividing both sides by  $f(K_a)$ , the above calculations give

$$h_N(t) = 1 - \frac{1}{2} \int_{a+}^t h_N(r) g(r) dr + \frac{\varepsilon_N^{(1)}(t) + \varepsilon_N^{(2)}(t)}{f(K_a)}. \tag{4.33}$$

Since  $|f| = 1$  and  $|e^{ia} - 1 - ia + a^2/2| \leq |a|^3/6$ , we have by (4.28)

$$\begin{aligned} |\varepsilon_N^{(2)}(t)| &\leq \frac{1}{6} \mathbb{E} \left[ \sum_{0 < r \leq T} |\Delta K_r|^3 | \mathcal{F}_a^N \right] \leq \frac{\|\phi\|}{6\sqrt{N}} \mathbb{E} \left[ \sum_{0 < r \leq T} (\Delta K_r)^2 | \mathcal{F}_a^N \right] \\ &\leq \frac{\|\phi\|}{6\sqrt{N}} \mathbb{E} [ [K]_T | \mathcal{F}_a^N ]. \end{aligned}$$

Since  $\mathbb{E}[K]_T = \int_0^T \mathbb{E}[g_N(s)] ds \rightarrow \int_0^T g(s) ds$ , we get  $\lim_{N \rightarrow \infty} \mathbb{E} \left[ \sup_{t \in [a, T]} |\varepsilon_N^{(2)}(t)| \right] = 0$ .

For  $\varepsilon_N^{(1)}$ , we let  $\psi(r) := Q_{(r, T)} \phi$  and  $\psi_N(r) = Q_{(r, T)}^N \phi$  for simplification. Since  $|f| = 1$ , the triangle inequality yields

$$\begin{aligned}
 & 2 \sup_{t \in [a, T]} \left| \varepsilon_N^{(1)}(t) \right| \\
 \leq & \mathbb{E} \left[ \int_{a+}^T \left| \langle \mathbf{a} \nabla \psi \cdot \nabla \psi, u(r) \rangle_\rho - \langle \mathbf{a} \nabla \psi \cdot \nabla \psi, \mathfrak{X}_r^N \rangle \right| dr \middle| \mathcal{F}_a^N \right] \\
 & + \mathbb{E} \left[ \int_{a+}^T \left| \langle \mathbf{a} \nabla \psi \cdot \nabla \psi - \mathbf{a} \nabla \psi_N \cdot \nabla \psi_N, \mathfrak{X}_r^N \rangle \right| dr \middle| \mathcal{F}_a^N \right] \\
 & + \int_{a+}^T \left| \int_{\partial D} \psi^2 q(r) u(r) \rho \, d\sigma - \langle \psi_N^2 q_N(r), u(r) \rangle_\rho \right| dr \\
 & + \sup_{t \in [a, T]} \left| \mathbb{E} \left[ \int_{a+}^t f(K_{r-}) \left( \langle \psi_N^2 q_N(r), u(r) \rangle_\rho - \langle \psi_N^2 q_N(r), \mathfrak{X}_r^N \rangle \right) dr \middle| \mathcal{F}_a^N \right] \right|. \tag{4.34}
 \end{aligned}$$

The expectation of the first term on the right-hand side of (4.34) tends to zero by the hydrodynamic result (Theorem 1.4). The expectation of the second term is at most

$$\begin{aligned}
 & \mathbb{E} \int_{a+}^T \langle |\mathbf{a} \nabla \psi \cdot \nabla \psi - \mathbf{a} \nabla \psi_N \cdot \nabla \psi_N|, \mathfrak{X}_r^N \rangle dr \\
 \leq & \int_{a+}^T \langle P_r(|\mathbf{a} \nabla \psi \cdot \nabla \psi - \mathbf{a} \nabla \psi_N \cdot \nabla \psi_N|), u_0 \rangle_\rho dr \\
 \leq & \|u_0\| \int_{a+}^T \langle |\mathbf{a} \nabla \psi \cdot \nabla \psi - \mathbf{a} \nabla \psi_N \cdot \nabla \psi_N|, 1 \rangle_\rho dr \\
 = & \|u_0\| \int_{a+}^T \langle |\mathbf{a} \nabla(\psi - \psi_N) \cdot \nabla(\psi + \psi_N)|, 1 \rangle_\rho dr \quad \text{by symmetry of } \mathbf{a} \\
 \leq & \|u_0\| \int_{a+}^T \sqrt{\mathcal{E}(\psi - \psi_N) \mathcal{E}(\psi + \psi_N)} \, dr \quad \text{by the Cauchy–Schwarz inequality.}
 \end{aligned}$$

This last quantity tends to zero as  $N \rightarrow \infty$  by Lemma 3.7 and Lebesgue dominated convergence theorem.

The third term (which is deterministic) on the right-hand side of (4.34) converges to zero as  $N \rightarrow \infty$  by Lemma 3.1 and the uniform convergence (3.8).

(iii) It remains to show that the forth (and last) term on the right-hand side of (4.34) converges to zero in  $L^1(\mathbb{P})$ . This term can be written as

$$\sup_{t \in [a, T]} \left| \mathbb{E} \left[ \int_{a+}^t f(K_{r-}) dH_N(r) - \int_{a+}^t f(K_{r-}) dG_N(r) \middle| \mathcal{F}_a^N \right] \right|, \tag{4.35}$$

where

$$H_N(t) := \int_{a+}^t \langle \psi_N^2 q_N(r), \mathfrak{X}_r^N \rangle dr \quad \text{and} \quad G_N(t) := \int_{a+}^t \langle \psi_N^2 q_N(r), u(r) \rangle_\rho dr.$$

We have by [Lemma 3.3](#)

$$\lim_{N \rightarrow \infty} \mathbb{E} \left[ \left( \sup_{t \in [0, T]} |H_N(t) - G_N(t)| \right)^p \right] = 0 \quad \text{for every } p \geq 1. \tag{4.36}$$

In view of [\(4.32\)](#) and [\(4.28\)](#), it suffice to show [\(4.35\)](#) converges to zero in  $L^1(\mathbb{P})$  with  $f(K_t)$  replaced by  $\tilde{f}(K_t) := 1 + \int_{0+}^t f'(K_{r-}) dK_r + \frac{1}{2} \int_{0+}^t f''(K_{r-}) d[K]_r$ . Furthermore, since  $H_N(t)$  and  $G_N(t)$  have bounded variations, by an integration by parts (see, e.g., Corollary 2 in [\[25, Chapter II\]](#)), we have

$$\begin{aligned} \int_{a+}^t \tilde{f}(K_{r-}) dH_N(r) &= \tilde{f}(K_t) H_N(t) - \int_{a+}^t H_N(r) d\tilde{f}(K_r) \quad \text{and} \\ \int_{a+}^t \tilde{f}(K_{r-}) dG_N(r) &= \tilde{f}(K_t) G_N(t) - \int_{a+}^t G_N(r) d\tilde{f}(K_r). \end{aligned}$$

On subtraction, it suffices to show

$$\mathbb{E} \left[ \sup_{t \in [a, T]} |\tilde{f}(K_t) (H_N(t) - G_N(t))| \middle| \mathcal{F}_a^N \right] \quad \text{and} \tag{4.37}$$

$$\sup_{t \in [a, T]} \left| \mathbb{E} \left[ \int_{a+}^t (H_N(r) - G_N(r)) f(K_r) d[K]_r \middle| \mathcal{F}_a^N \right] \right| \tag{4.38}$$

both converge to zero in  $L^1(\mathbb{P})$ .

Since  $|f| = 1$ ,  $|e^{ia} - 1 - ia + a^2/2| \leq |a|^3/6$ , we have by [\(4.28\)](#) and [\(4.32\)](#),

$$\sup_{t \in [a, T]} |\tilde{f}(K_{t-})| \leq \sup_{t \in [a, T]} (|f(K_{t-})| + |f(K_{t-}) - \tilde{f}(K_{t-})|) \leq 1 + \frac{\|\phi\| [K]_T}{6\sqrt{N}}.$$

Hence [\(4.37\)](#) converges to zero in  $L^1(\mathbb{P})$  by [\(4.30\)](#) and [\(4.36\)](#). Finally, the expectation of [\(4.38\)](#) is at most

$$\begin{aligned} & \mathbb{E} \left[ \sup_{t \in [a, T]} |(H_N(r) - G_N(r))| ([K]_T - [K]_a) \right] \\ & \leq \left( \mathbb{E} \left[ \sup_{t \in [a, T]} (H_N(r) - G_N(r))^2 \right] \right)^{1/2} (\mathbb{E} [[K]_T^2])^{1/2}, \end{aligned}$$

which goes to 0 as  $N \rightarrow \infty$  by (4.30) and (4.36). Hence by (4.34),  $\sup_{t \in [a, T]} |\varepsilon_N^{(1)}(t)| \rightarrow 0$  in  $L^1(\mathbb{P})$ . We then conclude from (4.33) that (4.31) holds. The proof of the lemma is now complete.  $\square$

We can now present the proof of Theorem 4.9.

**Proof of Theorem 4.9.** For notational convenience, set  $J_N(t) := \int_0^t \mathbf{U}_{(t,r)}^N dM_r^N$  and  $J(t) := \int_0^t \mathbf{U}_{(t,r)} dM_r$ .

(i) **Continuity of the limit.** In Appendix A, we checked that  $J(t)$  is a well-defined  $\mathcal{H}_{-\alpha}$ -valued Gaussian random variable. We now prove that  $J(\cdot)$  has a version in  $C^\gamma([0, T], \mathcal{H}_{-\alpha})$  for any  $\gamma \in (0, 1/2)$ . By Kolmogorov continuity criteria, it suffices to show that for  $\alpha > d + 2$  and  $n \in \mathbb{N}$ ,

$$\mathbb{E} \left[ |J(t) - J(s)|_{-\alpha}^{2n} \right] \leq C (t - s)^n \quad \text{whenever } 0 \leq s \leq t \leq T, \tag{4.39}$$

where  $C = C(n, d, D, T, \alpha) \|u_0\|^n (1 \vee \|q\|)^{4n} > 0$  is a constant.

Note that for  $\phi \in C(\overline{D})$ ,

$$\langle J(t) - J(s), \phi \rangle = \left\langle \int_s^t \mathbf{U}_{(t,r)} dM_r, \phi \right\rangle + \left\langle \int_0^s \mathbf{U}_{(t,r)} - \mathbf{U}_{(s,r)} dM_r, \phi \right\rangle,$$

which, as the sum of two independent centered Gaussian variables, is a centered Gaussian random variable with variance

$$V_s^t(\phi_k) := \int_s^t \mathcal{E}_r^{(q)}(Q_{(r,t)}\phi_k) dr + \int_0^s \mathcal{E}_r^{(q)}(Q_{(r,t)}\phi_k - Q_{(r,s)}\phi_k) dr.$$

By Lemma 3.9, we have

$$\begin{aligned} & \mathbb{E} [\langle J(t) - J(s), \phi_k \rangle^{2n}] = (2n - 1)!! (V_s^t(\phi_k))^n \\ & \leq C(n, d, D, T) \|u_0\|^n (1 \vee \|q\|)^{4n} (1 \vee \lambda_k)^{2n} \|\phi_k\|^{2n} (t - s)^n \end{aligned}$$

for any  $0 \leq s \leq t \leq T$  and  $k \in \mathbb{N}$ . Applying Hölder inequality  $(\sum_i a_i b_i)^n \leq (\sum_i a_i^{n/(n-1)})^{n-1} (\sum_i b_i^n)$ , we have for any  $\beta \in (0, \alpha]$ ,

$$\begin{aligned}
 & \mathbb{E} [|J(t) - J(s)|_{-\alpha}^{2n}] \\
 &= \mathbb{E} \left[ \left( \sum_k (1 + \lambda_k)^{-\alpha} \langle J(t) - J(s), \phi_k \rangle^2 \right)^n \right] \\
 &\leq \left( \sum_k (1 + \lambda_k)^{-\frac{\beta n}{n-1}} \right)^{n-1} \left( \sum_k (1 + \lambda_k)^{-(\alpha-\beta)n} \mathbb{E} [\langle J(t) - J(s), \phi_k \rangle^{2n}] \right) \\
 &\leq C \left( \sum_k \frac{1}{(1 + \lambda_k)^{\frac{\beta n}{n-1}}} \right)^{n-1} \left( \sum_k \frac{(1 \vee \lambda_k)^{2n} \|\phi_k\|^{2n}}{(1 + \lambda_k)^{(\alpha-\beta)n}} \right) (t - s)^n.
 \end{aligned}$$

It follows from (2.9) that (4.40) holds true if we choose  $\beta \in \left(\frac{d(n-1)}{2n}, \alpha - \frac{d}{2} - 2 - \frac{d}{2n}\right)$ . This choice of  $\beta$  is possible if and only if  $\alpha > 2 + d$ .

(ii) **Tightness.** We will show that there exists  $N_0 = N_0(D)$  such that for  $\alpha > d + 2$ ,

$$\sup_{N > N_0} \mathbb{E} [|J_N(t) - J_N(s)|_{-\alpha}^4] \leq C (t - s)^2 \tag{4.40}$$

whenever  $0 \leq s \leq t \leq T$ , where  $C = C(d, D, T, \alpha, \|u_0\|, \|q\|) \in (0, \infty)$  is a constant independent of  $N, s$  and  $t$ . By the Kolmogorov–Centov tightness criteria (see [12, Theorem 3.8.8]), (4.40) implies tightness of  $\{J_N\}_{N \geq 1}$  in  $D([0, T], \mathcal{H}_{-\alpha})$ .

Using Hölder inequality  $(\sum_i a_i b_i)^n \leq (\sum_i a_i^{n/(n-1)})^{n-1} (\sum_i b_i^n)$  (with  $n = 2$ ) and the condition  $\alpha > d + 2$  as in step (i) above, it suffices to show that

$$\sup_{N \geq N_0} \mathbb{E} [\langle J_N(t) - J_N(s), \phi_k \rangle^4] \leq C (1 \vee \lambda_k)^4 \|\phi_k\|^4 (t - s)^2 \tag{4.41}$$

for any  $0 \leq s \leq t \leq T$  and  $k \in \mathbb{N}$ , where  $N_0 = N_0(D)$  and  $C = C(d, D, T, \|u_0\|, \|q\|)$ .

We now prove (4.41) by first writing

$$J_N(t) - J_N(s) = \left( \int_0^s \mathbf{U}_{(t,r)}^N - \mathbf{U}_{(s,r)}^N dM_r^N \right) + \int_s^t \mathbf{U}_{(t,r)}^N dM_r^N.$$

Fix  $\phi_k$  and  $s \leq t$ . Observe that

$$\Gamma_w := \left\langle \int_0^w \mathbf{U}_{(t,r)}^N - \mathbf{U}_{(s,r)}^N dM_r^N, \phi_k \right\rangle$$

is a martingale for  $w \in [0, s]$ . As in (4.28), the jump size  $\Delta \Gamma_w := \Gamma_w - \Gamma_{w-}$  satisfies

$$\sup_{w \in [0, s]} |\Delta \Gamma_w| \leq \sup_{r \in [0, s]} \|Q_{r,t}^N \phi_k - Q_{r,s}^N \phi_k\| / \sqrt{N}.$$

Moreover, by Theorem 4.3, the dual predictable projection  $\langle \Gamma \rangle$  of the quadratic variation  $[\Gamma]$  of  $\Gamma$  is

$$\langle \Gamma \rangle_w = \int_0^w \mathcal{E}_r^{(q),N}(Q_{(r,t)}^N \phi_k - Q_{(r,s)}^N \phi_k) dr \quad \text{for } w \in [0, s],$$

where  $\mathcal{E}_r^{(q),N}(\phi, \psi) := \langle \mathbf{a} \nabla \phi \cdot \nabla \psi + q_N(r) \phi \psi, \mathfrak{X}_r^N \rangle$  and  $\mathcal{E}_r^{(q),N}(\phi) := \mathcal{E}_r^{(q),N}(\phi, \phi)$ . Therefore, by Burkholder–Davis–Gundy inequality for discontinuous martingales (see the remark after Theorem 74 in Chapter IV of [25]), we have  $\mathbb{E}[\Gamma_s^4] \leq \bar{c} \mathbb{E}[\langle \Gamma \rangle_s^2]$  for some absolute constant  $\bar{c}$ . Hence, argue as in (4.30), and then by Lemma 3.9 and Lemma 3.8, we obtain

$$\begin{aligned} & \mathbb{E}[\Gamma_s^4] \leq \bar{c} \mathbb{E}[\langle \Gamma \rangle_s^2] \\ & \leq 2\bar{c} \left( \mathbb{E}[\langle \Gamma \rangle_s^2] + \frac{\sup_{r \in [0,s]} \|Q_{r,t}^N \phi_k - Q_{r,s}^N \phi_k\|^2}{N} \mathbb{E}[\langle \Gamma \rangle_s] \right) \\ & \leq 2\bar{c} \mathbb{E}[\langle \Gamma \rangle_s^2] + C \frac{\|\phi_k\|^2 (\lambda_k^2 (t-s)^2 + (t-s))}{N} (\lambda_k^2 + \|\phi_k\|^2 + \lambda_k^2 \|\phi_k\|^2) (t-s), \end{aligned}$$

where  $C = C(D, T, \|u_0\|, \|q\|) \in (0, \infty)$ . Estimating  $\mathbb{E}[\langle \Gamma \rangle_s^2]$  by the argument we used for  $H_N^k(t)$  in the proof of Lemma 3.3 (via an inequality in Remark 3.10), we see that

$$\mathbb{E} \left[ \left\langle \int_0^s \mathbf{U}_{(t,r)}^N - \mathbf{U}_{(s,r)}^N dM_r^N, \phi_k \right\rangle^4 \right] = \mathbb{E}[(\Gamma_s)^4]$$

is bounded above by the RHS of (4.41) for  $N \geq N_0(D)$ .

Similarly, by consider the martingale

$$\Theta_w := \left\langle \int_s^{s+w} \mathbf{U}_{(t,r)}^N dM_r^N, \phi_k \right\rangle, \quad w \in [0, t-s];$$

and by using Lemma 3.9, we can check that  $\mathbb{E} \left[ \left\langle \int_s^t \mathbf{U}_{(t,r)}^N dM_r^N, \phi_k \right\rangle^4 \right] = \mathbb{E}[(\Theta_{t-s})^4]$  is bounded above by the RHS of (4.41) for  $N \geq N_0(D)$ .

**(iii) Convergence of finite dimensional distributions.** As in the proof of Theorem 4.8, it suffices to show that as  $N \rightarrow \infty$ ,

$$(\langle J_N(t_1), \psi_1 \rangle, \dots, \langle J_N(t_n), \psi_n \rangle) \xrightarrow{\mathcal{L}} (\langle J(t_1), \psi_1 \rangle, \dots, \langle J(t_n), \psi_n \rangle) \quad \text{in } \mathbb{R}^n \quad (4.42)$$

for any  $n \in \mathbb{N}$ ,  $0 \leq t_1 \leq \dots \leq t_n < \infty$  and  $\{\psi_j\}_{j=1}^n \subset C(\bar{D})$ .

For  $n = 1$ , fix  $t \geq 0$  and  $\phi \in C(\bar{D})$ . Note that  $\theta \mapsto \langle \int_0^\theta \mathbf{U}_{(t,s)}^N dM_s^N, \phi \rangle$  is a martingale for  $\theta \in [0, t]$ , with jumps size at most  $\sup_{\theta \in [0,t]} |M_\theta^N(Q_{(\theta,t)}^N \phi) - M_{\theta-}^N(Q_{(\theta,t)}^N \phi)| \leq \|\phi\|/\sqrt{N}$ , by (3.9) and (4.18). Hence by the functional central limit theorem for real-valued martingales (see [22]),

$$\left\{ \left\langle \int_0^\theta \mathbf{U}_{(t,s)}^N dM_s^N, \phi \right\rangle; \theta \in [0, t] \right\} \xrightarrow{\mathcal{L}} \left\{ \left\langle \int_0^\theta \mathbf{U}_{(t,s)} dM_s, \phi \right\rangle; \theta \in [0, t] \right\} \quad \text{in } D([0, t], \mathbb{R}) \tag{4.43}$$

as  $N \rightarrow \infty$ .

For an integer  $n > 1$ , (4.42) follows from Lemma 4.10 and the towering property of conditional expectations,

$$\mathbb{E}Z = \mathbb{E}\mathbb{E}[Z|\mathcal{F}_{t_1}] = \mathbb{E}\mathbb{E}[\mathbb{E}[Z|\mathcal{F}_{t_2}]|\mathcal{F}_{t_1}] = \dots \quad \text{for } 0 \leq t_1 \leq t_2 \leq t_3 \leq \dots$$

We illustrate this for the case  $n = 3$ ; the proof for the general case is the same. Observe that

$$\begin{aligned} \mathbb{E} \left[ \exp \left( i \sum_{k=1}^3 a_k \langle J_{t_k}^N, \psi_k \rangle \right) \right] &= \mathbb{E} \left[ \exp \left( i \sum_{j=1}^3 a_j \left\langle \int_0^{t_1} \mathbf{U}_{(t_j,s)}^N dM_s^N, \psi_j \right\rangle \right) \right. \\ &\quad \cdot \mathbb{E} \left[ \exp \left( i \sum_{j=2}^3 a_j \left\langle \int_{t_1}^{t_2} \mathbf{U}_{(t_j,s)}^N dM_s^N, \psi_j \right\rangle \right) \right. \\ &\quad \left. \left. \cdot \mathbb{E} \left[ \exp \left( i a_3 \left\langle \int_{t_2}^{t_3} \mathbf{U}_{(t_3,s)}^N dM_s^N, \phi_3 \right\rangle \right) \middle| \mathcal{F}_{t_2} \right] \middle| \mathcal{F}_{t_1} \right] \right]. \end{aligned}$$

We then apply Lemma 4.10 three times successively, starting from the inner most term involving  $\mathcal{F}_{t_2}$ . Hence we have convergence (4.42).

The proof of the lemma is complete.  $\square$

### 4.5. Characterization of $\mathcal{Y}$

Let  $\mathcal{Y}$  be any subsequential limit of  $\mathcal{Y}^N$ . By Theorem 4.7,  $\mathcal{Y}$  has a continuous version in  $\mathcal{H}_{-\alpha}$  for every  $\alpha > d \vee (d/2 + 2)$ . It follows from Theorems 4.3, 4.8 and 4.9 that we have

$$\mathcal{Y}_t \stackrel{\mathcal{L}}{=} \mathbf{U}_{(t,0)} \mathcal{Y}_0 + \int_0^t \mathbf{U}_{(t,s)} dM_s, \quad \text{in } D([0, T], \mathcal{H}_{-\alpha}). \tag{4.44}$$

**Theorem 4.11.** *The limiting process  $\mathcal{Y}$  is a continuous Gaussian Markov process that is unique in distribution. Moreover,  $\mathcal{Y}$  has a version in  $C^\gamma([0, \infty), \mathcal{H}_{-\alpha})$  for  $\gamma \in (0, 1/2)$ .*

**Proof.** Since  $M$  is Gaussian,  $\int_0^t \mathbf{U}_{(t,s)} dM_s$  is a Gaussian process by the construction of the stochastic integral. On the other hand,  $\mathbf{U}_{(t,0)} \mathcal{Y}_0$  is a Gaussian process and is



independent of  $\int_0^t \mathbf{U}_{(t,s)} dM_s$  since  $M$  has independent increments. Therefore  $\mathcal{Y}_t$ , as the sum of two independent Gaussian processes, is a Gaussian process.

The Markov property of  $\mathcal{Y}$  is basically due to the independent increments of the differentials; see Section 5.6 of [25]. For reader’s convenience, we give a proof that  $\mathcal{Y}$  is a Markov process with respect to its own filtration  $\mathcal{F}_t^{\mathcal{Y}} := \sigma(\mathcal{Y}_r : r \leq t) = \sigma(\langle \mathcal{Y}_r, \phi \rangle : r \leq t, \phi \in \mathcal{H}_\alpha)$ . We in particular have from (4.44) that for  $s \leq t$ ,

$$\mathcal{Y}_t \stackrel{\mathcal{L}}{=} \mathbf{U}_{(t,s)} \mathcal{Y}_s + \int_s^t \mathbf{U}_{(t,r)} dM_r \quad \text{in } \mathcal{H}_{-\alpha}. \tag{4.45}$$

Together with the fact that  $M$  has independent increments, we have

$$\text{Cov}(\langle \mathcal{Y}_s, \phi \rangle, \langle \mathcal{Y}_t, \psi \rangle) = \text{Cov}(\langle \mathcal{Y}_s, \phi \rangle, \langle \mathbf{U}_{(t,s)} \mathcal{Y}_s, \psi \rangle) \tag{4.46}$$

for all  $s \leq t$  and  $\phi, \psi \in \mathcal{H}_\alpha$ . To show that  $\mathcal{Y}$  is Markov, note that (4.46) together with the fact that  $\mathbf{U}_{(t,s)} \mathcal{Y}_s \in \mathcal{F}_s^{\mathcal{Y}}$  yield  $\mathbb{E}[F(\mathcal{Y}_t) | \mathcal{F}_s^{\mathcal{Y}}] = F(\mathbf{U}_{(t,s)} \mathcal{Y}_s)$  for all  $F \in C_b(\mathcal{H}_{-\alpha})$ . Using (4.45) and the fact that  $\mathbf{U}_{(t,s)} \mathcal{Y}_s \in \sigma(\mathcal{Y}_s)$ , we obtain  $\mathbb{E}[F(\mathcal{Y}_t) | \mathcal{Y}_s] = F(\mathbf{U}_{(t,s)} \mathcal{Y}_s)$  for all  $F \in C_b(\mathcal{H}_{-\alpha})$ . This shows that  $\mathcal{Y}$  is Markov.

The Hölder continuity of  $\mathcal{Y}$  follows immediately from Theorem 4.8 and Theorem 4.9.  $\square$

The proof of Theorem 1.5 is now complete.

## Appendix A

### A.1. Hilbert–Schmidt operators

Hilbert–Schmidt operators appear naturally in stochastic analysis in infinite dimensions. The main properties of these operators can be found in standard references (e.g. [13]). We now recall the main definitions.

**Definition A.1.** Let  $X = (X_t)_{t \geq 0}$  be an  $\mathcal{H}_{-\alpha}$ -valued process defined on a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ . We say  $X$  is (centered) **Gaussian** if  $\{X_t(\phi) : \phi \in \mathcal{H}_\alpha, t \in [0, \infty)\}$  form a (centered) Gaussian system. That is,  $(X_{t_1}(\phi_1), \dots, X_{t_k}(\phi_k))$  is a (centered) Gaussian vector in  $\mathbb{R}^k$  for any  $k \in \mathbb{N}$ , any  $\{t_i\}_{i=1}^k \subset [0, \infty)$  and any  $\{\phi_i\}_{i=1}^k \subset \mathcal{H}_\alpha$ . We say  $X$  is **continuous** if  $t \mapsto X_t$  is continuous  $\mathbb{P}$ -a.s.  $X$  is said to be **square-integrable** if  $\mathbb{E}[|X_t|_{-\alpha}^2] < \infty$  for all  $t \geq 0$ . Finally, we say  $X$  has **independent increments** if for any  $0 \leq s < t$  and  $\phi \in \mathcal{H}_\alpha$ , the real random variable  $X_t(\phi) - X_s(\phi)$  is independent of the  $\sigma$ -field generated by  $\{X_r(\psi) : 0 \leq r \leq s, \psi \in \mathcal{H}_\alpha\}$ .

Suppose  $X$  and  $Y$  are real separable Hilbert spaces with inner product  $\langle \cdot, \cdot \rangle_X$  and  $\langle \cdot, \cdot \rangle_Y$  (we simply write  $\langle \cdot, \cdot \rangle$  when there is no confusion for which Hilbert space we are

considering). The class of bounded linear operators from  $X$  to  $Y$  will be denoted by  $L(X, Y)$  ( $L(X)$  for short when  $X = Y$ ). It is well known that  $A \in L(X, Y)$  is **compact** (i.e. the range of the unit sphere in  $X$  is relatively compact in  $Y$ ) if and only if there exist orthonormal systems (ONS for short)  $\{e_n\} \subset X$ ,  $\{f_n\} \subset Y$  and a sequence of real numbers  $a_n \rightarrow 0$  such that  $A$  has the representation

$$Ax = \sum_{n \geq 1} a_n \langle x, e_n \rangle f_n \quad \text{for all } x \in X. \tag{A.1}$$

**Definition A.2.**

1.  $A \in L(X, Y)$  is said to be **Hilbert–Schmidt** (denoted by  $A \in L_2(X, Y)$ ) if  $A$  has the representation (A.1) with  $\sum_{n \geq 1} a_n^2 < \infty$ . In this case, the **Hilbert–Schmidt norm** of  $A$  is defined to be

$$\|A\|_2 := \left( \sum_{n \geq 1} a_n^2 \right)^{1/2} = \left( \sum_{n \geq 1} |Ae_n|^2 \right)^{1/2}$$

Note that  $\|A\|_2$  is independent of the choice of the ONS  $\{e_n\} \subset X$ .

2. The **Trace** of  $A \in L(X)$  is

$$Tr(A) := \sum_{n \geq 1} \langle Ae_n, e_n \rangle$$

Note that  $Tr(A)$  is independent of the choice of the ONS  $\{e_n\} \subset X$ .

The following lemma is equivalent to the statement that  $(\Phi_{imb}, \mathcal{H}_\beta, \mathcal{H}_\gamma)$  is an abstract Wiener space if  $\beta > d/2 + \gamma$  (cf. [26]).

**Lemma A.3.** *For any  $\beta, \gamma \in \mathbb{R}$  with  $\beta > \gamma + d/2$ , the imbedding  $\Phi_{imb} : \mathcal{H}_\beta \rightarrow \mathcal{H}_\gamma$  is Hilbert–Schmidt.*

**Proof.** We want to show that  $\sum_k \left| \Phi_{imb} \left( h_k^{(\beta)} \right) \right|_\gamma^2 < \infty$ . The left-hand side equals

$$\sum_k (1 + \mu_k)^{-\beta} |\phi_k|_\gamma^2 = \sum_k (1 + \lambda_k)^{-\beta + \gamma}.$$

By Weyl’s formula (2.8), the latter quantity is finite if and only if

$$\int_1^\infty (1 + x)^{-\beta + \gamma} x^{d/2 - 1} dx < \infty.$$

This is true if and only if  $\beta - \gamma > d/2$ .  $\square$

A.2.  $\int_0^t \mathbf{U}_{(t,s)} dM_s$  is well defined

As mentioned earlier, we have to make sure that  $\mathbf{U}_{(t,s)}$  (for  $s \in [0, t]$ ) lies within the class of integrands with respect to  $M$ . We will follow the construction of stochastic integrals with respect to Hilbert space valued r.c.l.l. square-integrable martingales in [23]. See [9,13,25] for more comprehensive and recent treatments.

We denote by  $M_c^2([0, \infty), \mathcal{H}_{-\alpha})$  the class of continuous square-integrable  $\mathcal{H}_{-\alpha}$ -valued martingales with zero initial value on a complete probability space  $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$  equipped with right continuous  $\{\tilde{\mathcal{F}}_t\}$ . Fix  $\alpha > d \vee (d/2 + 1)$  and recall from Theorem 4.6 that  $M \in M_c^2([0, \infty), \mathcal{H}_{-\alpha})$  is Gaussian, has independent increments and covariance

$$\tilde{\mathbb{E}}[\langle M_s, \phi \rangle \langle M_t, \psi \rangle] = \int_0^{s \wedge t} \mathcal{E}_r^{(q)}(\phi, \psi) dr, \tag{A.2}$$

where  $\mathcal{E}_r^{(q)}$  is the bilinear form on  $\mathcal{H}_{-\alpha}$  defined in (1.10). We will omit the filtration when there is no ambiguity. For example, we simply say that  $M$  is adapted rather than  $\tilde{\mathcal{F}}_t$ -adapted since it is defined on  $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathcal{F}}_t, \tilde{\mathbb{P}})$ . For  $T \in (0, \infty]$ , denote by  $\mathfrak{P}_{[0,T]}$  the  $\sigma$ -field of predictable sets on  $\tilde{\Omega} \times [0, T]$ . That is, the smallest  $\sigma$ -field making all adapted processes with left continuous paths measurable (cf. p. 156 of [25] or Section 1.7 of [23]). When  $T = \infty$ , we write  $\mathfrak{P}$  for  $\mathfrak{P}_{[0,\infty)}$ .

By a direct calculation,

$$[[M]]_t := \sum_k \int_0^t \mathcal{E}_r^{(q)}(h_k^{(\alpha)}) dr \tag{A.3}$$

is the unique continuous, adapted and increasing real process such that  $|M_t|_{-\alpha}^2 - [[M]]_t$  is a real martingale (cf. Remark 2.2 in [13]).  $[[M]]$  is called the **real increasing process** associated to  $M$ . Besides, the operators  $Q_s : H_{-\alpha} \rightarrow H_{-\alpha}$  (for  $s \geq 0$ ) defined by

$$\langle Q_s(h_i^{(-\alpha)}), h_j^{(-\alpha)} \rangle_{-\alpha} := \frac{\mathcal{E}_s^{(q)}(h_i^{(\alpha)}, h_j^{(\alpha)})}{\sum_k \mathcal{E}_s^{(q)}(h_k^{(\alpha)})} \tag{A.4}$$

is called the **characteristic operator process** associated to  $M$ . Clearly,  $Q_s$  is a non-negative operator on  $H_{-\alpha}$  with  $Tr(A) = 1$  where ‘Tr’ means ‘Trace’. As a remark, the operator-valued process  $\langle\langle M \rangle\rangle_t := \int_0^t Q_s d[[M]]s$  (in the sense of Bochner’s integral) is called the **operator increasing process associated to  $M$**  and plays an analogous role as the quadratic variation of real-valued martingales (see Theorem 2.3 in Chapter 1 of [13] for its basic properties).

Following [23], the class of possible integrands for the stochastic integral with  $M_t$  as integrator (on the interval  $[0, T]$ ) can be defined as follows: On the space of  $\mathfrak{P}_{[0,T]}$ -simple  $L(\mathcal{H}_{-\alpha})$ -valued processes, we define a scalar product

$$(A, B) := \tilde{\mathbb{E}} \left[ \int_0^T \text{Tr}(A Q_s B^*) d[[M]]_s \right], \tag{A.5}$$

where  $B^*$  is the adjoint of the operator  $B$ . The completion of the  $\mathfrak{P}_{[0,T]}$ -simple  $L(\mathcal{H}_{-\alpha})$ -valued processes with respect to the scalar product in (A.5), denoted by  $\Lambda^2(\mathcal{H}_{-\alpha}, \mathfrak{P}_{[0,T]}, M)$ , is the desired class of integrands. It is worth noting that (cf. p. 171 [23])  $\Lambda^2(\mathcal{H}_{-\alpha}, \mathfrak{P}_{[0,T]}, M)$  contains processes whose values may be unbounded operators.

By Section 1.3 of [13],  $\Lambda^2(\mathcal{H}_{-\alpha}, \mathfrak{P}_{[0,T]}, M)$  contains the class of all processes  $(\Phi_t)_{t \in [0,T]}$  such that

- (i)  $\Phi_t$  is a linear operator (not necessarily bounded) from  $\sqrt{Q_t} \mathcal{H}_{-\alpha}$  to  $\mathcal{H}_{-\alpha}$  such that  $\Phi_t \sqrt{Q_t} \in L_2(\mathcal{H}_{-\alpha})$  is Hilbert–Schmidt for all  $t \in [0, T]$  a.s.,
- (ii)  $\Phi_t \sqrt{Q_t}$  is  $\mathfrak{P}|_{\tilde{\Omega} \times [0,T]}$ -measurable (i.e. predictable), and
- (iii)  $\mathbb{E} \left[ \int_0^T \|\Phi_t \sqrt{Q_t}\|_2^2 d[[M]]_t \right] < \infty$  where  $\|\cdot\|_2$  is the Hilbert–Schmidt norm.

Now for any  $t > 0$ , the deterministic process  $(U_{(t,\theta)})_{\theta \in [0,t]}$  lies in the class of integrands with respect to  $M$ . This is because on one hand

$$\begin{aligned} \|U_{(t,\theta)} \sqrt{Q_\theta}\|_2^2 &= \text{Tr} \left( U_{(t,\theta)} Q_\theta U_{(t,\theta)}^* \right) \quad \text{the trace of } U_{(t,\theta)} Q_\theta U_{(t,\theta)}^* \\ &= \sum_k \langle U_{(t,\theta)} Q_\theta U_{(t,\theta)}^* (h_k^{(-\alpha)}), h_k^{(-\alpha)} \rangle_{-\alpha} \\ &= \sum_k \langle Q_\theta U_{(t,\theta)}^* (h_k^{(-\alpha)}), U_{(t,\theta)}^* (h_k^{(-\alpha)}) \rangle_{-\alpha} \\ &= \frac{\sum_k \mathcal{E}_\theta^{(q)}(Q_{(\theta,t)} h_k^{(\alpha)})}{\sum_i \mathcal{E}_\theta^{(q)}(h_i^{(\alpha)})} \end{aligned}$$

which is finite provided that  $u_0$  is not identically zero; and on the other hand, by Lemma 3.9,

$$\begin{aligned} &\mathbb{E} \left[ \int_0^t \|U_{(t,\theta)} \sqrt{Q_\theta}\|_2^2 d[[M]]_\theta \right] \\ &= \sum_k \int_0^t \mathcal{E}_\theta^{(q)}(Q_{(\theta,t)} h_k^{(\alpha)}) d\theta \\ &\leq C(d, D, T) \|u_0\| \left( t \sum_k \frac{\lambda_k + \|\phi_k^2\|}{(1 + \lambda_k)^\alpha} + \|q\| t^{3/2} \sum_k \frac{\|\phi_k\|^2}{(1 + \lambda_k)^\alpha} \right) \quad \text{for } t \in [0, T] \\ &< \infty \quad \text{if } \alpha > d \vee (d/2 + 1). \end{aligned}$$

We conclude that for any fixed  $t \geq 0$ ,  $\{\int_0^s U_{(t,\theta)} dM_\theta; s \in [0, t]\}$  is a continuous, adapted square-integrable  $\mathcal{H}_{-\alpha}$ -valued martingale with  $\mathbb{E} \left[ \left| \int_0^s U_{(t,\theta)} dM_\theta \right|_{-\alpha}^2 \right] = \sum_k \int_0^s \mathcal{E}_\theta^{(q)}(Q_{(\theta,t)} h_k^{(\alpha)}) d\theta$ . In particular, putting  $s = t$ , we have that  $\int_0^t U_{(t,\theta)} dM_\theta$  is a well defined  $\tilde{\mathcal{F}}_t$ -measurable  $\mathcal{H}_{-\alpha}$ -valued random variable with finite second moment. Moreover, since  $M$  is centered Gaussian with independent increments,  $\int_0^t U_{(t,\theta)} dM_\theta$  is also centered Gaussian.

A.3. An identity

The following equality is used in Lemma 3.8.

Lemma A.4.

$$\int_0^s \cdots \int_0^{s_k} \frac{1}{\sqrt{(s-s_2)(s_2-s_3)\cdots(s_k-s_{k+1})}} ds_{k+1} \cdots ds_2 = \frac{\pi^{k/2}}{\Gamma(\frac{k+2}{2})} s^{k/2}.$$

Proof. Denote the integral on the left-hand side as  $V_k$ . For any  $a \in (0, \infty)$ ,

$$\int_0^x \frac{y^a}{\sqrt{x-y}} dy = \frac{\sqrt{\pi} \Gamma(1+a)}{\Gamma(3/2+a)} x^{1/2+a}$$

Using this, we can iterate it to obtain  $V_k = \int_0^t c_k s^{k/2} ds$ , where

$$c_1 = 2 \quad \text{and} \quad c_{k+1} = c_k \frac{\sqrt{\pi} \Gamma(1+k/2)}{\Gamma(3/2+k/2)} \text{ for } k \geq 2. \quad \square$$

A.4. Reflected diffusions killed by local time

Suppose now, instead of being killed by  $q_N$ , that  $Z_i^{(N)} = Z_i$  is the subprocess of  $X_i$  killed by  $2 \int_0^t q(s, X_i(s)) dL_s^{(i)}$  for all  $N$  and all  $1 \leq i \leq N$ . In Remark 1.7(ii), we claimed that Theorem 1.4 and Theorem 1.5 remain valid. The claim that Theorem 1.4 remains true is easy to be verified. We now provide some details to support the claim that Theorem 1.5 remains valid.

By the same proof of Lemma 3.12, we have the following:

Lemma A.5. Fix any positive integer  $N$ . For any  $\phi \in \text{Dom}^{\text{Feller}}(\mathcal{A})$ , we have under  $\mathbb{P}^\mu$  for any  $\mu \in E_N$ ,

$$M_t^\phi := \langle \phi, \mathfrak{X}_t^N \rangle - \langle \phi, \mathfrak{X}_0^N \rangle - \int_0^t \langle \mathcal{A}\phi, \mathfrak{X}_s^N \rangle ds + \frac{1}{N} \sum_{i=1}^N \int_0^t q(s, Z_i(s)) \phi(Z_i(s)) dL_s^i$$

is an  $\mathcal{F}_t^{\mathfrak{X}^N}$ -martingale under  $\mathbb{P}^\mu$  for any  $\mu \in E_N$ . Moreover,  $M_t^\phi$  has predictable quadratic variation

$$\langle M^\phi \rangle_t = \frac{1}{N} \left[ \int_0^t \langle \mathbf{a} \nabla \phi \cdot \nabla \phi, \mathfrak{X}_s^N \rangle ds + \frac{1}{N} \sum_{i=1}^N \int_0^t q(s, Z_i(s)) \phi^2(Z_i(s)) dL_s^i \right].$$

Moreover, (3.26) still holds for this new martingale.

Starting from the above lemma, we just need slight modifications in the proof of Theorem 1.5. It is easier in this case since now we have  $Q^N = Q$  and  $\mathbf{U}^N = \mathbf{U}$ . Note that in the proof of Lemma 4.10, the expressions  $H_N(t)$  and  $G_N(t)$  in (4.35) should be replaced by, respectively,

$$\frac{1}{N} \sum_{i=1}^N \int_0^t \psi^2(Z_r^i) q(r, Z_r^i) dL_r^i \quad \text{and} \quad \int_0^t \int_{\partial D} \psi^2(z) q(r, z) u(r, z) \rho(z) d\sigma(z) dr.$$

In addition, we should also use the following lemma rather than Lemma 3.3.

**Lemma A.6.** *Let  $\{\phi(r) : r \geq 0\} \subset C(\overline{D})$  be such that  $\sup_{r \geq 0} \|\phi(r)\| < \infty$ . For any  $p \geq 1$ , we have*

$$\lim_{N \rightarrow \infty} \mathbb{E} \left[ \left( \sup_{t \in [0, T]} \left| \sum_{i=1}^N \int_0^t \phi(r, Z_r^i) dL_r^i - \int_0^t \int_{\partial D} \phi(r, z) u(r, z) \rho(z) d\sigma(z) dr \right|^p \right) \right] = 0.$$

The proof of Lemma A.6 is the same as that of Lemma 3.3.

### References

- [1] R.F. Bass, P. Hsu, Some potential theory for reflecting Brownian motion in Hölder and Lipschitz domains, *Ann. Probab.* 19 (1991) 486–508.
- [2] C. Boldrighini, A. De Masi, A. Pellegrinotti, Nonequilibrium fluctuations in particle systems modelling reaction–diffusion equations, *Stochastic Process. Appl.* 42 (1992) 1–30.
- [3] Z.-Q. Chen, On reflecting diffusion processes and Skorokhod decompositions, *Probab. Theory Related Fields* 94 (1993) 281–316.
- [4] Z.-Q. Chen, W.-T. Fan, Hydrodynamic limits and propagation of chaos for interacting random walks in domains, Preprint, arXiv:1311.2325 [math.PR].
- [5] Z.-Q. Chen, W.-T. Fan, Systems of interacting diffusions with partial annihilations through membranes, *Ann. Probab.* (2015), <http://dx.doi.org/10.1214/15-AOP1047>, in press.
- [6] Z.-Q. Chen, W.-T. Fan, Fluctuation limits for interacting diffusions with partial annihilations through membranes, Preprint, arXiv:1410.5905 [math.PR].
- [7] Z.-Q. Chen, M. Fukushima, *Symmetric Markov Processes, Time Change and Boundary Theory*, Princeton University Press, 2012.
- [8] R.F. Curtain, H. Zwart, *An Introduction to Infinite-Dimensional Linear Systems Theory*, vol. 21, Springer, Berlin, 1995.
- [9] G. Da Prato, J. Zabczyk, *Stochastic Equations in Infinite Dimensions*, Cambridge University Press, 2008.

- [10] P. Dittrich, A stochastic particle system: fluctuations around a nonlinear reaction–diffusion equation, *Stochastic Process. Appl.* 30 (1988) 149–164.
- [11] R. Durrett, *Probability: Theory and Examples*, Cambridge University Press, 2010.
- [12] S.N. Ethier, T.G. Kurtz, *Markov Processes. Characterization and Convergence*, Wiley, New York, 1986, MR0838085.
- [13] W. Grecksch, C. Tudor, *Stochastic Evolution Equations: A Hilbert Space Approach*, Akad.-Verlag, Berlin, 1995.
- [14] P. Gyrya, L. Saloff-Coste, Neumann and Dirichlet Heat Kernels in Inner Uniform Domains, *Astérisque*, vol. 336, 2011, viii+144 pp.
- [15] M. Hairer, An introduction to stochastic PDEs, arXiv:0907.4178, 2009.
- [16] K. Itô, Continuous additive S'-processes, in: B. Frigielionis (Ed.), *Stochastic Differential Systems*, Springer, Berlin, 1980.
- [17] K. Itô, Distribution-valued processes arising from independent Brownian motions, *Math. Z.* 182 (1983) 17–33.
- [18] I. Karatzas, S.E. Shreve, *Brownian Motion and Stochastic Calculus*, Graduate Texts in Mathematics, Springer, 1991.
- [19] P. Kotelenez, A submartingale type inequality with applications to stochastic evolution equations, *Stochastics* 8 (1982) 139–151.
- [20] P. Kotelenez, Law of large numbers and central limit theorem for linear chemical reactions with diffusion, *Ann. Probab.* 14 (1986) 173–193.
- [21] P. Kotelenez, High density limit theorems for nonlinear chemical reactions with diffusion, *Probab. Theory Related Fields* 78 (1988) 11–37.
- [22] R.S. Lipcer, A.N. Shirayev, A functional central limit theorem for semimartingales, *Teor. Veroyatn. Primen.* 25 (1980) 638–703 (English translation in *Theory Probab. Appl.*).
- [23] M. Metivier, J. Pellaumail, *Stochastic Integration*, Academic, New York, London, 1980.
- [24] Y. Netrusov, Y. Safarov, Weyl asymptotic formula for the Laplacian on domains with rough boundaries, *Comm. Math. Phys.* 253 (2005) 481–509.
- [25] P.E. Protter, *Stochastic Integration and Differential Equations*, 2nd ed., Springer, Berlin, 2005.
- [26] S. Sheffield, Gaussian free fields for mathematicians, *Probab. Theory Related Fields* 139 (2007) 521–541.
- [27] P. Stollmann, J. Voigt, Perturbation of Dirichlet forms by measures, *Potential Anal.* 5 (2) (1996) 109–138.
- [28] A.S. Sznitman, Nonlinear reflecting diffusion process, and the propagation of chaos and fluctuations associated, *J. Funct. Anal.* 56 (1984) 311–336.

